MATH (Science)

NOES

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Unit 01

MATRICES AND DETERMINANTS

Idea of matrices:

The idea of matrices was given by Arthur Cayley, an English mathematician of nineteenth century who first developed, "Theory of Matrices" in 1858.

Q1. Define the following terms.

(i) Matrix

"A rectangular array or a formation of a collection of real numbers, say 0, 1, 2, 3, 4 and 7, such as: $\begin{matrix}1 & 3 & 4\\ 7 & 2 & 0\end{matrix}$ and then enclosed by brackets '[ ]' is said to form a matrix $\begin{bmatrix}1 & 3 & 4\\ 7 & 2 & 0\end{bmatrix}$. Similarly $\begin{bmatrix}0 & 1\\ 3 & 4\end{bmatrix}$ is another matrix.

The matrices are denoted conventionally by the capital letters A,B,C,....,M,N etc. of the English alphabet.

(ii) Order of a Matrix

The number of rows and columns in a matrix specifies its order. If a matrix M has m rows and n columns then M is said to be of order, m-by-n, For example, $M=\begin{bmatrix}1 & 2 & 3\\ 1 & 0 & 2\end{bmatrix}$ is order 2-by-3,

(iii) Equal Matrices

Let A and B be two matrices. Then A is said to be equal to B, and is denoted by A = B, if and only if;

(i) The order of A = The order of B

(ii) Their corresponding entries are equal.

Examples

(i) $A=\begin{bmatrix}1 & 3\\ -4 & 2\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2+1\\ -4 & 4-2\end{bmatrix}$

are equal matrices.

We see that:

(a) The order of matrix A = The order of matrix B

(b) Their corresponding elements are equal.

Thus A = B

(ii) $L=\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}$ and $M=\begin{bmatrix}2 & 3\\ -1 & -2\end{bmatrix}$ are not equal matrices.

We see that: order of L = order of M but entries in the second row and second column are not same, so $L \neq M$.

(iii) $P=\begin{bmatrix}2 & 3\\ -1 & 2\end{bmatrix}$ and $Q=\begin{bmatrix}2 & 3 & 4\\ -1 & 2 & 0\end{bmatrix}$

are not equal matrices.

We see that order of P ≠ order of Q, so $P \neq Q$.

Exercise 1.1

1. Find the order of the following matrices.

$A=\begin{bmatrix}2 & 3\\ -5 & 6\end{bmatrix}$, order of A is 2-by-2

$B=\begin{bmatrix}2 & 0\\ 3 & 5\end{bmatrix}$, order of B is 2-by-2

$C=\begin{bmatrix}2 & 4\end{bmatrix}$ order of C is 1-by-2

$D=\begin{bmatrix}4\\ 0\\ 6\end{bmatrix}$, order of D is 3-by-1

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- کوئی بھی ممبر کسی بھی ممبر کو انباکس میں میسج، مس کال، کال نہیں کرے گا۔ رپورٹ پر فوری ریموو کرکے کاروائی عمل میں لائے جائے گی۔
- ہمارے کسی بھی گروپ میں سیاسی و فرقہ واریت کی بحث کی قطعاً کوئی گنجائش نہیں ہے۔
- اگر کسی کو بھی گروپ کے متعلق کسی قسم کی شکایت یا تجویز کی صورت میں ایڈ من سے رابطہ کیجئے۔
- سب سے اہم بات:

گروپ میں کسی بھی قادیانی، مرزائی، احمدی، گستاخِ رسول، گستاخِ امہات المؤمنین، گستاخِ صحابہ و خلفائے راشدین حضرت ابو بکر صدیق، حضرت عمر فاروق، حضرت عثمان غنی، حضرت علی المرتضیٰ، حضرت حسنین کریمین رضوان اللہ تعالیٰ اجمعین، گستاخ اہلبیت یا ایسے غیر مسلم جو اسلام اور پاکستان کے خلاف پراپیگنڈا میں مصروف ہیں یا ان کے روحانی و ذہنی سپورٹرز کے لئے کوئی گنجائش نہیں ہے لہذا ایسے اشخاص بالکل بھی گروپ جوائن کرنے کی زحمت نہ کریں۔ معلوم ہونے پر فوراً ریموو کر دیا جائے گا۔

- تمام کتب انٹرنیٹ سے تلاش / ڈاؤنلوڈ کر کے **فری آف کاسٹ** وٹس ایپ گروپ میں شیئر کی جاتی ہیں۔ جو کتاب نہیں ملتی اس کے لئے معذرت کر لی جاتی ہے۔ جس میں محنت بھی صَرف ہوتی ہے لیکن ہمیں آپ سے صرف دعاؤں کی درخواست ہے۔
- عمران سیریز کے شوقین کیلئے علیحدہ سے عمران سیریز گروپ موجود ہے۔
- **لیڈیز کے لئے الگ گروپ کی سہولت موجود ہے جس کے لئے ویریفکیشن ضروری ہے۔**
- اردو کتب / عمران سیریز یا سٹڈی گروپ میں ایڈ ہونے کے لئے ایڈ من سے وٹس ایپ پر بذریعہ میسج رابطہ کریں اور جواب کا انتظار فرمائیں۔ برائے مہربانی اخلاقیات کا خیال رکھتے ہوئے موبائل پر کال یا ایم ایس کرنے کی کوشش ہر گز نہ کریں۔ ورنہ گروپس سے تو ریموو کیا ہی جائے گا بلاک بھی کیا جائے گا۔

نوٹ: ہمارے کسی گروپ کی کوئی فیس نہیں ہے۔ سب فی سبیل اللہ ہے

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اللہ تبارک تعالیٰ ہم سب کا حامی و ناصر ہو

$$E=\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix},$$ order of E is 3-by-2

$$F=[2]$$ order of F is 1-by-1

$$G=\begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix},$$ order of G is 3-by-3

$$H=\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 6 \end{bmatrix}$$ order of H is 2-by-3

2. Which of the following matrices are equal?

$A=[3]$,

$B=[3 \quad 5], C=[5 - 2]$

$D=[5 \quad 3], E=\begin{bmatrix} 4 & 0 \\ 6 & 2 \end{bmatrix},$

$F=\begin{bmatrix} 2 \\ 6 \end{bmatrix}$

$G=\begin{bmatrix} 3-1 \\ 3+3 \end{bmatrix},$

$H=\begin{bmatrix} 4 & 0 \\ 6 & 2 \end{bmatrix}, I=[3 \quad 3+2]$

$J=\begin{bmatrix} 2+2 & 2-2 \\ 2+4 & 2+0 \end{bmatrix}$

Ans. Equal matrices are

$A = C \qquad B = I$

$E = H = J \qquad F = G$

3. Find the values of a, b, c and d which satisfy the matrix equation.

$$\begin{bmatrix} a+c & a+2b \\ c-1 & 4d-6 \end{bmatrix}=\begin{bmatrix} 0 & -7 \\ 3 & 2d \end{bmatrix}$$

Ans. $a + c = 0$ ....(i)

$a + 2b = -7$ ....(ii)

$c - 1 = 3$ ....(iii)

$4d - 6 = 2d$ ......(iv)

From (iii)

$c = 3+1$

$\boxed{c=4}$

From (iv)

$4d - 2d = 6$

$2d = 6$

$d=\frac{6}{2}$

$\boxed{d=3}$

Put value of c = 4 in (i)j

$a + 4 = 0$

$\boxed{a=-4}$

Put value of a = –4 in (ii)

$-4 + 2b = -7$

$2b = -7 + 4$

$2b = -3$

$\boxed{b=\frac{-3}{2}}$

Types of Matrices

(e) **Row Matrix.**

A matrix is called a row matrix if it has only one row.

e.g.; the matrix $M=[2 \quad -1 \quad 7]$ is a row matrix of order 1-by-3 and $M=[1 \quad -1]$ is a row matrix of order 1-by-2.

(ii) **Column Matrix.**

A matrix is called a column matrix if it has only one column e.g., $M=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

and $N = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ are column matrices of order 2-by-1 and 3-by-1 respectively.

(e) **Rectangular Matrix.**

A matrix is called rectangular if, the number of rows of M is not equal to the number of columns of M.

e.g. $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$;

$B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$; $C = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ and

$D = \begin{bmatrix} 7 \\ 8 \\ 0 \end{bmatrix}$ are all rectangular matrices. The order of A is 3-by-2, the order of B is 2-by-3, the order of C is 1-by-3 and order of D is 3-by-1, which indicates that in each matrix the number of rows $\neq$ the number of columns.

(e) **Square Matrix.**

A matrix is called a square matrix if its number of rows is equal to its number of columns.

e.g., $A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 \end{bmatrix}$ are square matrices of orders 2-by-2, 3-by-3 and 1-by-1 respectively.

(v) **Null or Zero Matrix.**

A matrix M is called a null or zero matrix if each of its entries is 0.

e.g., $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are null matrices of orders 2-by-2, 1-by-2, 2-by-1, 2-by-3 and 3-by-3 respectively. Null matrix is represented by O.

(vi) **Transpose of a Matrix.**

A matrix obtained by interchanging the rows into columns or columns into rows of a matrix is called transpose of that matrix. If A is a matrix then its transpose is denoted by A^t.

e.g., (i) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ -1 & 4 & -2 \end{bmatrix}$, then $A^t = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ 3 & 0 & -2 \end{bmatrix}$

Note: *If a matrix A is of order 2-by-3 then order of its transpose A^t is 3-by-2*

(vii) **Negative of a Matrix.**

Let A be a matrix. Then its negative, $-A$, is obtained by changing the signs of all the entries of A, i.e.,

If $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, then $-A = \begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$.

(viii) **Symmetric Matrix.**

A square matrix is symmetric if it is equal to its transpose i.e., matrix A is symmetric if $A^t = A$.

e.g. (i) If $M=\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 4 & 0\end{bmatrix}$

is a square matrix, then

$M^t=\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 4\\ 3 & 4 & 0\end{bmatrix}=M$. Thus M is a symmetric matrix.

(ii) If $A=\begin{bmatrix}2 & 1 & 3\\ -1 & 2 & 2\\ 3 & 1 & 3\end{bmatrix}$,

then $A^t=\begin{bmatrix}2 & -1 & 3\\ 1 & 2 & 1\\ 3 & 2 & 3\end{bmatrix}\neq A$

Hence A is not a symmetric matrix.

(x) Skew-Symmetric Matrix.

A square matrix A is said to be skew-symmetric if $A^t=-A$.

e.g., If $A=\begin{bmatrix}0 & 2 & 3\\ -2 & 0 & 1\\ -3 & -1 & 0\end{bmatrix}$, then

$$A^t=\begin{bmatrix}0 & -2 & -3\\ 2 & 0 & -1\\ 3 & 1 & 0\end{bmatrix}=\begin{bmatrix}0 & -2 & -3\\ -(-2) & 0 & -1\\ -(-3) & -(-1) & 0\end{bmatrix}$$

$$=-\begin{bmatrix}0 & 2 & 3\\ -2 & 0 & 1\\ -3 & -1 & 0\end{bmatrix}=-A$$

Since $A^t=-A$, therefore A is a skew-symmetric matrix.

(x) Diagonal Matrix.

A square matrix A is called a diagonal matrix if atleast any one of the entries of its diagonal is not zero and non-diagonal entries must all be zero.

e.g., $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$, $B=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$

and $C=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 3\end{bmatrix}$ are all diagonal matrices of order 3-by-3.

$M=\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$ and $N=\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}$

are diagonal matrices of order 2-by-2.

(xi) Scalar Matrix.

A diagonal matrix is called a scalar matrix, if all the diagonal entries are same and non-zero. For example $\begin{bmatrix}k & 0 & 0\\ 0 & k & 0\\ 0 & 0 & k\end{bmatrix}$

where k is a constant $\neq 0, 1$.

Also $A=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$, $B=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$ and

$C=[5]$ are scalar matrices of order 3-by-3, 2-by-2 and 1-by-1 respectively.

(xii) Identity Matrix.

A diagonal matrix is called identity (unit) matrix if all diagonal entries are 1 and it is denoted by I.

e.g., $A=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$ is a 3-by-3 identity matrix.

$B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ is a 2-by-2 identity matrix.

$C=[1]$ is a 1-by-1 identity matrix.

Exercise 1.2

1. From the following matrices, identify unit matrices, row matrices, column matrices and null matrices.

Ans. $A=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, Null matrix

$B=\begin{bmatrix}2 & 3 & 4\end{bmatrix}$, Row matrix

$C=\begin{bmatrix}4\\ 0\\ 6\end{bmatrix}$, Column matrix

$D=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, Unit matrix

$E=\begin{bmatrix}0\end{bmatrix}$, Null matrix

$F=\begin{bmatrix}5\\ 6\\ 7\end{bmatrix}$ Column matrix

2. From the following matrices, identify

(a) Square matrices
(b) Rectangular matrices
(c) Row matrices
(d) Column matrices
(e) Identity matrices
(f) Null matrices

Ans. (a) Square Matrices:

(iii) $\begin{bmatrix}6 & -4\\ 3 & -2\end{bmatrix}$

(iv) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

(viii) $\begin{bmatrix}1 & 2 & 3\\ -1 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$

Ans. (b) Rectangular Matrices:

(i) $\begin{bmatrix}-8 & 2 & 7\\ 12 & 0 & 4\end{bmatrix}$

(ii) $\begin{bmatrix}3\\ 0\\ 1\end{bmatrix}$

(v) $\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}$

Ans. (c) Row Matrices:

(vi) $\begin{bmatrix}3 & 10 & -1\end{bmatrix}$

Ans. (d) Column Matrices:

(ii) $\begin{bmatrix}3\\ 0\\ 1\end{bmatrix}$

(vii) $\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$

Ans. (e) Identity Matrices:

(iv) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Ans. (f) Null matrices:

(ix) $\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}$

3. From the following matrices, identify diagonal, scalar and unit (identity) matrices.

$$A=\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix},$$

$$B=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix},$$

$$C=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

$$D=\begin{bmatrix}3 & 0\\ 0 & 0\end{bmatrix},$$

$$E=\begin{bmatrix}5-3 & 0\\ 0 & 1+1\end{bmatrix}$$

Ans. Scalar matrices:

$$A=\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix},\ E=\begin{bmatrix}5-3 & 0\\ 0 & 1+1\end{bmatrix}$$

Unit Matrices:

$$C=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Diagonal Matrices:

$$A=\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix},\ B=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix},$$

$$C=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},D=\begin{bmatrix}3 & 0\\ 0 & 0\end{bmatrix}$$

$$E=\begin{bmatrix}5-3 & 0\\ 0 & 1+1\end{bmatrix}$$

4. Find negative of matrices A, B, C, D and E when:

$$A=\begin{bmatrix}1\\ 0\\ -1\end{bmatrix},$$

$$B=\begin{bmatrix}3 & -1\\ 2 & 1\end{bmatrix}$$

$$C=\begin{bmatrix}2 & 6\\ 3 & 2\end{bmatrix},\ D=\begin{bmatrix}-3 & 2\\ -4 & 5\end{bmatrix},$$

$$E=\begin{bmatrix}1 & -5\\ 2 & 3\end{bmatrix}$$

Negative of matrices

Ans. $-A=\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix},$

Ans. $-B=\begin{bmatrix}-3 & 1\\ -2 & -1\end{bmatrix}$

Ans. $-C=\begin{bmatrix}-2 & -6\\ -3 & -2\end{bmatrix}$

Ans. $-D=\begin{bmatrix}3 & -2\\ 4 & -5\end{bmatrix},$

Ans. $E=\begin{bmatrix}-1 & 5\\ -2 & -3\end{bmatrix}$

5. Find the transpose of each of the following matrices:

Ans. (i)

$$A=\begin{bmatrix}0\\ 1\\ -2\end{bmatrix},\Rightarrow\ A^t=\begin{bmatrix}0 & 1 & -2\end{bmatrix}$$

$$B=\begin{bmatrix}5 & 1 & -6\end{bmatrix}\Rightarrow B^t=\begin{bmatrix}5\\ 1\\ -6\end{bmatrix}$$

$$C=\begin{bmatrix}1 & 2\\ 2 & -1\\ 3 & 0\end{bmatrix}\quad\Rightarrow\quad C^t=\begin{bmatrix}1 & 2\\ 2 & -1\end{bmatrix}$$

$D=\begin{bmatrix}2 & 3\\0 & 5\end{bmatrix} \Rightarrow D^t=\begin{bmatrix}2 & 0\\3 & 5\end{bmatrix}$

$E=\begin{bmatrix}2 & 3\\-4 & 5\end{bmatrix} \Rightarrow E^t=\begin{bmatrix}2 & -4\\3 & 5\end{bmatrix}$

$F=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix} \Rightarrow F^t=\begin{bmatrix}1 & 3\\2 & 4\end{bmatrix}$

6. Verify that if $A=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$, $B=\begin{bmatrix}1 & 1\\2 & 0\end{bmatrix}$, **then**

(i) $(A^t)^t = A$

(ii) $(B^t)^t = B$

Ans. (i) $(A^t)^t = A$

$L.H.S = (A^t)^t$

$A=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$

$\Rightarrow A^t=\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}$

$(A^t)^t=\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$

$(A^t)^t = A = R.H.S.$

Hence L.H.S = R.H.S.

Ans. (ii) $(B^t)^t = B$

$L.H.S = (B^t)^t$

$B=\begin{bmatrix}1 & 1\\2 & 0\end{bmatrix}$

$B^t=\begin{bmatrix}1 & 2\\1 & 0\end{bmatrix}$

$(B^t)^t=\begin{bmatrix}1 & 1\\2 & 0\end{bmatrix}$

$(B^t)^t = B$

$= R.H.S$

Hence L.H.S = R.H.S.

Addition and Subtraction of Matrices

Define Addition of Matrices.

Let A and B be any two matrices with real number entries. The matrices A and B are conformable for addition, if they have the same order.

e.g., $A=\begin{bmatrix}2 & 3 & 0\\1 & 0 & 6\end{bmatrix}$ and $B=\begin{bmatrix}-2 & 3 & 4\\1 & 2 & 3\end{bmatrix}$ are conformable for addition.

Addition of A and B, written A+ B is obtained by adding the entries of the matrix A to the corresponding entries of the matrix B.

e.g., $A+B=\begin{bmatrix}2 & 3 & 0\\1 & 0 & 6\end{bmatrix}+\begin{bmatrix}-2 & 3 & 4\\1 & 2 & 3\end{bmatrix}$

$=\begin{bmatrix}2+(-2) & 3+3 & 0+4\\1+1 & 0+2 & 6+3\end{bmatrix}=\begin{bmatrix}0 & 6 & 4\\2 & 2 & 9\end{bmatrix}$

Define Subtraction of Matrices.

If A and B are two matrices of same order then subtraction of matrix B from matrix A is obtained by subtracting the entries of matrix B from the corresponding entries of matrix A and it is denoted by A – B.

e.g., $A=\begin{bmatrix}2 & 3 & 4\\1 & 5 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & 2 & 2\\-1 & 4 & 3\end{bmatrix}$ are conformable for subtraction.

i.e., $A-B=\begin{bmatrix}2 & 3 & 4\\1 & 5 & 0\end{bmatrix}-\begin{bmatrix}0 & 2 & 2\\-1 & 4 & 3\end{bmatrix}$

$=\begin{bmatrix}2-0 & 3-2 & 4-2\\1-(-1) & 5-4 & 0-3\end{bmatrix}=\begin{bmatrix}2 & 1 & 2\\2 & 1 & -3\end{bmatrix}$

Multiplication of a Matrix by a Real Number

Let A be any matrix and the real number k be a scalar. Then the scalar

multiplication of matrix A with k is obtained by multiplying each entry of matrix A with k. It is denoted by kA.

Let $A=\begin{bmatrix}1 & -1 & 4\\ 2 & -1 & 0\\ -1 & 3 & 2\end{bmatrix}$ be a matrix of order 3-by-3 and k=–2 be a real number.

Then

$$kA=(-2)A$$

$$=(-2)\begin{bmatrix}1 & -1 & 4\\ 2 & -1 & 0\\ -1 & 3 & 2\end{bmatrix}$$

$$=\begin{bmatrix}(-2)1 & (-2)(-1) & (-2)(4)\\ (-2)(2) & (-2)(-1) & (-2)(0)\\ (-2)(-1) & (-2)(3) & (-2)(2)\end{bmatrix}$$

$$=\begin{bmatrix}-2 & 2 & -8\\ -4 & 2 & 0\\ 2 & -6 & -4\end{bmatrix}$$

Commutative and Associative Laws of Matrices

(a) Commutative Law under Addition

If A and B are two matrices of the same order, then A + B = B + A is called commutative law under addition.

Let $A=\begin{bmatrix}2 & 3 & 0\\ 5 & 6 & 1\\ 2 & 1 & 3\end{bmatrix}$,

$$B=\begin{bmatrix}3 & -2 & 5\\ -1 & 4 & 1\\ 4 & 2 & -4\end{bmatrix}$$

Then

$$A+B=\begin{bmatrix}2 & 3 & 0\\ 5 & 6 & 1\\ 2 & 1 & 3\end{bmatrix}+\begin{bmatrix}3 & -2 & 5\\ -1 & 4 & 1\\ 4 & 2 & -4\end{bmatrix}$$

$$=\begin{bmatrix}2+3 & 3-2 & 0+5\\ 5-1 & 6+4 & 1+1\\ 2+4 & 1+2 & 3-4\end{bmatrix}=\begin{bmatrix}5 & 1 & 5\\ 4 & 10 & 2\\ 6 & 3 & -1\end{bmatrix}$$

Similarly

$$B+A=\begin{bmatrix}3 & -2 & 5\\ -1 & 4 & 1\\ 4 & 2 & -4\end{bmatrix}+\begin{bmatrix}2 & 3 & 0\\ 5 & 6 & 1\\ 2 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}5 & 1 & 5\\ 4 & 10 & 2\\ 6 & 3 & -1\end{bmatrix}$$

Thus the commutative law of addition of matrices is verified.

A + B = B + A

(b) Associative Law under Addition

If A, B and C are three matrices of same order, such that (A+B)+C=A+(B+C) is called associative law under addition.

Let $A=\begin{bmatrix}2 & 3 & 0\\ 5 & 6 & 1\\ 2 & 1 & 3\end{bmatrix}$,

$$B=\begin{bmatrix}3 & -2 & 5\\ -1 & 4 & 1\\ 4 & 2 & -4\end{bmatrix}$$

and

$$C=\begin{bmatrix}1 & 2 & 3\\ -2 & 0 & 4\\ 1 & 2 & 0\end{bmatrix}$$

Then

$$(A+B)+C=\left(\begin{bmatrix}2&3&0\\5&6&1\\2&1&3\end{bmatrix}+\begin{bmatrix}3&-2&5\\-1&4&1\\4&2&-4\end{bmatrix}\right)$$

$$+\begin{bmatrix}1&2&3\\-2&0&4\\1&2&0\end{bmatrix}$$

$$=\begin{bmatrix}2+3&3-2&0+5\\5-1&6+4&1+1\\2+4&1+2&3-4\end{bmatrix}+\begin{bmatrix}1&2&3\\-2&0&4\\1&2&0\end{bmatrix}$$

$$=\begin{bmatrix}5&1&5\\4&10&2\\6&3&-1\end{bmatrix}+\begin{bmatrix}1&2&3\\-2&0&4\\1&2&0\end{bmatrix}$$

$$=\begin{bmatrix}6&3&8\\2&10&6\\7&5&-1\end{bmatrix}$$

$$A+(B+C)=\begin{bmatrix}2&3&0\\5&6&1\\2&1&3\end{bmatrix}$$

$$+\left(\begin{bmatrix}3&-2&5\\-1&4&1\\4&2&-4\end{bmatrix}+\begin{bmatrix}1&2&3\\-2&0&4\\1&2&0\end{bmatrix}\right)$$

$$=\begin{bmatrix}2&3&0\\5&6&1\\2&1&3\end{bmatrix}+\begin{bmatrix}3+1&-2+2&5+3\\-1-2&4+0&1+4\\4+1&2+2&-4+0\end{bmatrix}$$

$$=\begin{bmatrix}2&3&0\\5&6&1\\2&1&3\end{bmatrix}+\begin{bmatrix}4&0&8\\-3&4&5\\5&4&-4\end{bmatrix}$$

$$=\begin{bmatrix}6&3&8\\2&10&6\\7&5&-1\end{bmatrix}$$

Thus the associative law of addition is verified:

$$(A+B)+C = A+(B+C)$$

Additive Identity of a Matrix

If A and B are two matrices of same order such that $A + B = A = B + A$ then matrix B is called additive identity of matrix A.

For any matrix A and zero matrix O of same order, O is called additive identity of A as

$A + O = A = O + A$

e.g., let $A=\begin{bmatrix}1&2\\3&5\end{bmatrix}$ and $O=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

then

$$A+O=\begin{bmatrix}1&2\\3&5\end{bmatrix}+\begin{bmatrix}0&0\\0&0\end{bmatrix}=\begin{bmatrix}1&2\\3&5\end{bmatrix}=A$$

$$O+A=\begin{bmatrix}0&0\\0&0\end{bmatrix}+\begin{bmatrix}1&2\\3&5\end{bmatrix}=\begin{bmatrix}1&2\\3&5\end{bmatrix}=A$$

Additive Inverse of a Matrix

If A and B are two matrices of same order such that $A + B = O = B + A$ then A and B are called additive inverse of each other.

Additive inverse of any matrix A is obtained by changing the signs of all the non zero entries of A.

Let $A=\begin{bmatrix}1&2&1\\0&-1&-2\\3&1&0\end{bmatrix}$

then

$$B=(-A)=-\begin{bmatrix}1&2&1\\0&-1&-2\\3&1&0\end{bmatrix}=\begin{bmatrix}-1&-2&-1\\0&1&2\\-3&-1&0\end{bmatrix}$$

is additive inverse of A. It can be verified as:

$$A+B=\begin{bmatrix}1&2&1\\0&-1&-2\\3&1&0\end{bmatrix}+\begin{bmatrix}-1&-2&-1\\0&1&2\\-3&-1&0\end{bmatrix}$$

$$=\begin{bmatrix}(1)+(-1) & (2)+(-2) & (1)+(-1)\\ 0+0 & (-1)+(1) & (-2)+(2)\\ (3)+(-3) & (1)+(-1) & 0+0\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=O$$

$$B+A=\begin{bmatrix}-1&-2&-1\\0&1&2\\-3&-1&0\end{bmatrix}+\begin{bmatrix}1&2&1\\0&-1&-2\\3&1&0\end{bmatrix}$$

$$=\begin{bmatrix}(-1)+(1) & (-2)+(2) & (-1)+(1)\\ 0+0 & (1)+(-1) & (2)+(-2)\\ (-3)+(3) & (-1)+(1) & 0+0\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=O$$

Since $A + B = O = B + A$

Therefore B is additive inverse of A.

Exercise 1.3

1. Which of the following matrices are conformable for addition?

$A=\begin{bmatrix}2&1\\-1&3\end{bmatrix}$, $B=\begin{bmatrix}3\\1\end{bmatrix}$,

$C=\begin{bmatrix}1&0\\2&-1\\1&-2\end{bmatrix}$, $D=\begin{bmatrix}2+1\\3\end{bmatrix}$,

$E=\begin{bmatrix}-1&0\\1&2\end{bmatrix}$,

$F=\begin{bmatrix}3&2\\1+1&-4\\3+2&2+1\end{bmatrix}$ **Ans. (i)**

$A=\begin{bmatrix}2&1\\-1&3\end{bmatrix}$, and $E=\begin{bmatrix}-1&0\\1&2\end{bmatrix}$

are conformable for addition.

(ii) $B=\begin{bmatrix}3\\1\end{bmatrix}$ and $D=\begin{bmatrix}2+1\\3\end{bmatrix}$

are conformable for addition.

(iii) $C=\begin{bmatrix}1&0\\2&-1\\1&-2\end{bmatrix}$ and $F=\begin{bmatrix}3&2\\1+1&-4\\3+2&2+1\end{bmatrix}$

are conformable for addition.

2. Find the additive inverse of following matrices.

$A=\begin{bmatrix}2&4\\-2&1\end{bmatrix}$,

$B=\begin{bmatrix}1&0&-1\\2&-1&3\\3&-2&1\end{bmatrix}$, $C=\begin{bmatrix}4\\-2\end{bmatrix}$

$D=\begin{bmatrix}1&0\\-3&-2\\2&1\end{bmatrix}$,

$E=\begin{bmatrix}1&0\\0&1\end{bmatrix}$,

$F=\begin{bmatrix}\sqrt{3}&1\\-1&\sqrt{3}\end{bmatrix}$

Ans.

(i) $A=\begin{bmatrix}2&4\\-2&1\end{bmatrix}$

Additive inverse of Matrix A is

$-A=-\begin{bmatrix}2&4\\-2&1\end{bmatrix}\Rightarrow -A=\begin{bmatrix}-2&-4\\2&-1\end{bmatrix}$

(ii) $B=\begin{bmatrix}1 & 0 & -1\\ 2 & -1 & 3\\ 3 & -2 & 1\end{bmatrix}$

Additive inverse of Matrix B is

$$-B=-\begin{bmatrix}1 & 0 & -1\\ 2 & -1 & 3\\ 3 & -2 & 1\end{bmatrix}$$

$$-B=\begin{bmatrix}-1 & 0 & 1\\ -2 & 1 & -3\\ -3 & 2 & -1\end{bmatrix}$$

(iii) $C=\begin{bmatrix}4\\ -2\end{bmatrix}$

Additive inverse of Matrix C is

$$-C=-\begin{bmatrix}4\\ -2\end{bmatrix} \Rightarrow -C=\begin{bmatrix}-4\\ 2\end{bmatrix}$$

(iv) $D=\begin{bmatrix}1 & 0\\ -3 & -2\\ 2 & 1\end{bmatrix}$

Additive inverse of Matrix D is

$$-D=-\begin{bmatrix}1 & 0\\ -3 & -2\\ 2 & 1\end{bmatrix} \Rightarrow -D=\begin{bmatrix}-1 & 0\\ 3 & 2\\ -2 & -1\end{bmatrix}$$

(v) $E=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Additive inverse of Matrix E is

$$-E=-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \Rightarrow -E=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

(vi) $F=\begin{bmatrix}\sqrt{3} & 1\\ -1 & \sqrt{2}\end{bmatrix}$

Additive inverse of Matrix F is

$$-F=-\begin{bmatrix}\sqrt{3} & 1\\ -1 & \sqrt{2}\end{bmatrix} \Rightarrow -F=\begin{bmatrix}-\sqrt{3} & -1\\ 1 & -\sqrt{2}\end{bmatrix}$$

3. If $A=\begin{bmatrix}-1 & 2\\ 2 & 1\end{bmatrix}$, $B=\begin{bmatrix}1\\ -1\end{bmatrix}$, $C=\begin{bmatrix}1 & -1 & 2\end{bmatrix}$ $D=\begin{bmatrix}1 & 2 & 3\\ -1 & 0 & 2\end{bmatrix}$, **then find,**

(i) $A+\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ (ii) $B+\begin{bmatrix}-2\\ 3\end{bmatrix}$

(iii) $C+\begin{bmatrix}-2 & 1 & 3\end{bmatrix}$

(iv) $D+\begin{bmatrix}0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}$ (v) 2A

(vi) (−1)B (vii) (−2)C

(viii) 3D (ix) 3C

Ans. (i) $A+\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$

$$=\begin{bmatrix}-1 & 2\\ 2 & 1\end{bmatrix}+\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}-1+1 & 1+2\\ 2+1 & 1+1\end{bmatrix}=\begin{bmatrix}0 & 3\\ 3 & 2\end{bmatrix}$$

(ii) $B+\begin{bmatrix}-2\\ 3\end{bmatrix}=$

$$=\begin{bmatrix}1\\ -1\end{bmatrix}+\begin{bmatrix}-2\\ 3\end{bmatrix}=\begin{bmatrix}1-2\\ -1+3\end{bmatrix}=\begin{bmatrix}-1\\ 2\end{bmatrix}$$

(iii) $C+\begin{bmatrix}-2 & 1 & 3\end{bmatrix}$

$$=\begin{bmatrix}1 & -1 & 2\end{bmatrix}+\begin{bmatrix}-2 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1-2 & -1+1 & 2+3\end{bmatrix}=\begin{bmatrix}-1 & 0 & 5\end{bmatrix}$$

(iv) $D+\begin{bmatrix}0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 2 & 3\\ -1 & 0 & 2\end{bmatrix}+\begin{bmatrix}0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}$

$$=\begin{bmatrix}1+0 & 2+1 & 0+3\\ -1+2 & 0+0 & 2+1\end{bmatrix}=\begin{bmatrix}1 & 3 & 3\\ 1 & 0 & 3\end{bmatrix}$$

(v) $2A=2\begin{bmatrix}-1 & 2\\ 2 & 1\end{bmatrix}=\begin{bmatrix}-2 & 4\\ 4 & 2\end{bmatrix}$

(vi) $-1(B)=(-1)\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}-1\\ 1\end{bmatrix}$

(vii) $(-2)C=(-2)\begin{bmatrix}1 & -1 & 2\end{bmatrix}$

$$=\begin{bmatrix}-2 & 2 & -4\end{bmatrix}$$

(viii) $3D=3\begin{bmatrix}0 & 1 & 0\\ 2 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 3 & 0\\ 6 & 0 & 3\end{bmatrix}$

(ix) $3C = 3\begin{bmatrix}1 & -1 & 2\end{bmatrix} = \begin{bmatrix}3 & -3 & 6\end{bmatrix}$

4. Perform the indicated operations and simplify the following.

(i) $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}0 & 2\\3 & 0\end{bmatrix}+\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$

(ii) $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}0 & 2\\3 & 0\end{bmatrix}-\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$

(iii) $\begin{bmatrix}2 & 3 & 1\end{bmatrix}$

$+\left(\begin{bmatrix}1 & 0 & 2\end{bmatrix}-\begin{bmatrix}2 & 2 & 2\end{bmatrix}\right)$

(iv) $\begin{bmatrix}1 & 2 & 3\\-1 & -1 & -1\\0 & 1 & 2\end{bmatrix}+\begin{bmatrix}1 & 1 & 1\\2 & 2 & 2\\3 & 3 & 3\end{bmatrix}$

(v) $\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\3 & 1 & 2\end{bmatrix}+\begin{bmatrix}1 & 0 & -2\\-2 & -1 & 0\\0 & 2 & -1\end{bmatrix}$

(vi) $\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}+\begin{bmatrix}2 & 1\\1 & 0\end{bmatrix}+\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

Ans. **(i)** $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}0 & 2\\3 & 0\end{bmatrix}+\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$

$= \begin{bmatrix}1+0+1 & 0+2+1\\0+3+1 & 1+0+0\end{bmatrix} = \begin{bmatrix}2 & 3\\4 & 1\end{bmatrix}$

(ii) $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}0 & 2\\3 & 0\end{bmatrix}-\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$

$= \begin{bmatrix}1+0-1 & 0+2-1\\0+3-1 & 1+0-0\end{bmatrix}$

$= \begin{bmatrix}0 & 1\\2 & 1\end{bmatrix}$

(iii) $\begin{bmatrix}2 & 3 & 1\end{bmatrix}+\left(\begin{bmatrix}1 & 0 & 2\end{bmatrix}-\begin{bmatrix}2 & 2 & 2\end{bmatrix}\right)$

$= \begin{bmatrix}2\,3\,1\end{bmatrix}+\begin{bmatrix}1-2 & 0-2 & 2-2\end{bmatrix}$

$= \begin{bmatrix}2 & 3 & 1\end{bmatrix}+\begin{bmatrix}-1 & -2 & 0\end{bmatrix}$

$= \begin{bmatrix}2-1 & 3-2 & 1+0\end{bmatrix}$

$= \begin{bmatrix}1 & 1 & 1\end{bmatrix}$

(iv) $\begin{bmatrix}1 & 2 & 3\\-1 & -1 & -1\\0 & 1 & 2\end{bmatrix}+\begin{bmatrix}1 & 1 & 1\\2 & 2 & 2\\3 & 3 & 3\end{bmatrix}$

$= \begin{bmatrix}1+1 & 2+1 & 3+1\\-1+2 & -1+2 & -1+2\\0+3 & 1+3 & 2+3\end{bmatrix}$

$= \begin{bmatrix}2 & 3 & 4\\1 & 1 & 1\\3 & 4 & 5\end{bmatrix}$

(v) $\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\3 & 1 & 2\end{bmatrix}+\begin{bmatrix}1 & 0 & -2\\-2 & -1 & 0\\0 & 2 & -1\end{bmatrix}$

$= \begin{bmatrix}1+1 & 2+0 & 3-2\\2-2 & 3-1 & 1-0\\3+0 & 1+2 & 2-1\end{bmatrix}$

$= \begin{bmatrix}2 & 2 & 1\\0 & 2 & 1\\3 & 3 & 1\end{bmatrix}$

(vi) $\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}+\begin{bmatrix}2 & 1\\1 & 0\end{bmatrix}+\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

$= \begin{bmatrix}1+2+1 & 2+1+1\\0+1+1 & 1+0+1\end{bmatrix}$

$= \begin{bmatrix}4 & 4\\2 & 2\end{bmatrix}$

5. For the matrices

$A = \begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\1 & -1 & 0\end{bmatrix},$

$B = \begin{bmatrix}1 & -1 & 1\\2 & -2 & 2\\3 & 1 & 3\end{bmatrix}$ **and**

$$C=\begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 3\\ 1 & 1 & 2\end{bmatrix}$$ **verify the following rules.**

(i) $A+C=C+A$

(ii) $A+B=B+A$

(iii) $B+C=C+B$

(iv) $A+(B+A)=2A+B$

(v) $(C-B)+A=C+(A-B)$

(vi) $2A+B=A+(A+B)$

(vii) $(C-B)-A=(C-A)-B$

(viii) $(A+B)+C=A+(B+C)$

(ix) $A(B-C)=(A-C)+B$

(x) $2A+2B=2(A+B)$

Ans.

(i) $A+C=C+A$

L.H.S = A + C

$$=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\\ 1 & -1 & 0\end{bmatrix}+\begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 3\\ 1 & 1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}1-1 & 2+0 & 3+0\\ 2+0 & 3-2 & 1+3\\ 1+1 & -1+1 & 0+2\end{bmatrix}$$

$$=\begin{bmatrix}0 & 2 & 3\\ 2 & 1 & 4\\ 2 & 0 & 2\end{bmatrix}$$

R.H.S = C + A

$$=\begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 3\\ 1 & 1 & 2\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\\ 1 & -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}-1+1 & 0+2 & 0+3\\ 0+2 & -2+3 & 3+1\\ 1+1 & 1-1 & 0+2\end{bmatrix}$$

$$=\begin{bmatrix}0 & 2 & 3\\ 2 & 1 & 4\\ 2 & 0 & 2\end{bmatrix}$$

L.H.S = R.H.S

(ii) $A+B=B+A$

L.H.S = A + B

$$=\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\\ 1 & -1 & 0\end{bmatrix}+\begin{bmatrix}1 & -1 & 1\\ 2 & -2 & 2\\ 3 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1+1 & 2-1 & 3+1\\ 2+2 & 3-2 & 1+2\\ 1+3 & -1+1 & 0+3\end{bmatrix}$$

$$=\begin{bmatrix}2 & 1 & 4\\ 4 & 1 & 3\\ 4 & 0 & 3\end{bmatrix}$$

R.H.S = B + A

$$=\begin{bmatrix}1 & -1 & 1\\ 2 & -2 & 2\\ 3 & 1 & 3\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\\ 1 & -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}1+1 & -1+2 & 1+3\\ 2+2 & -2+3 & 2+1\\ 3+1 & 1-1 & 3+0\end{bmatrix}$$

$$=\begin{bmatrix}2 & 1 & 4\\ 4 & 1 & 3\\ 4 & 0 & 3\end{bmatrix}$$

L.H.S. = R.H.S

(iii) $B+C=C+B$

L.H.S = B + C

$$=\begin{bmatrix}1 & -1 & 1\\ 2 & -2 & 2\\ 3 & 1 & 3\end{bmatrix}+\begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 3\\ 1 & 1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}1-1 & -1+0 & 1+0\\ 2+0 & -2-2 & 2+3\\ 3+1 & 1+1 & 3+2\end{bmatrix}$$

$$=\begin{bmatrix}0 & -1 & 1\\2 & -4 & 5\\4 & 2 & 5\end{bmatrix}$$

R.H.S = C + B

$$=\begin{bmatrix}-1 & 0 & 0\\0 & -2 & 3\\1 & 1 & 2\end{bmatrix}+\begin{bmatrix}1 & -1 & 1\\2 & -2 & 2\\3 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-1+1 & 0-1 & 0+1\\0+2 & -2-2 & 3+2\\1+3 & 1+1 & 3+2\end{bmatrix}$$

$$=\begin{bmatrix}0 & -1 & 1\\2 & -4 & 5\\4 & 2 & 5\end{bmatrix}$$

L.H.S = R.H.S.

(iv) $A+(B+A)=2A+B$

L.H.S = A+ (B+A)

$$=\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\1 & -1 & 0\end{bmatrix}+\begin{bmatrix}2 & 1 & 4\\4 & 1 & 3\\4 & 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1+2 & 2+1 & 3+4\\2+4 & 3+1 & 1+3\\1+4 & -1+0 & 0+3\end{bmatrix}$$

$$=\begin{bmatrix}3 & 3 & 7\\6 & 4 & 4\\5 & -1 & 3\end{bmatrix}$$

R.H.S = 2A + B

$$=2\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\1 & -1 & 0\end{bmatrix}+\begin{bmatrix}1 & -1 & 1\\2 & -2 & 2\\3 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}2 & 4 & 6\\4 & 6 & 2\\2 & -2 & 0\end{bmatrix}+\begin{bmatrix}1 & -1 & 1\\2 & -2 & 2\\3 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}2+1 & 4-1 & 6+1\\4+2 & 6-2 & 2+2\\2+3 & -2+1 & 0+3\end{bmatrix}$$

$$=\begin{bmatrix}3 & 3 & 7\\6 & 4 & 4\\5 & -1 & 3\end{bmatrix}$$

L.H.S = R.H.S

(v) $(C-B)+A=C+(A-B)$

L.H.S. = (C–B) + A

$$C-B=\begin{bmatrix}-1 & 0 & 0\\0 & -2 & 3\\1 & 1 & 2\end{bmatrix}-\begin{bmatrix}1 & -1 & 1\\2 & -2 & 2\\3 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-1-1 & 0+1 & 0-1\\0-2 & -2+2 & 3-2\\1-3 & 1-1 & 2-3\end{bmatrix}$$

$$=\begin{bmatrix}-2 & 1 & -1\\-2 & 0 & 1\\-2 & 0 & -1\end{bmatrix}$$

$$(C-B)+A=\begin{bmatrix}-2 & 1 & -1\\-2 & 0 & 1\\-2 & 0 & -1\end{bmatrix}+\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\1 & -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}-2+1 & 1+2 & -1+3\\-2+2 & 0+3 & 1+1\\-2+1 & 0-1 & -1+0\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 3 & 2\\0 & 3 & 2\\-1 & -1 & -1\end{bmatrix}$$

R.H.S. = C + (A – B)

$$A-B=\begin{bmatrix}1 & 2 & 3\\2 & 3 & 1\\1 & -1 & 0\end{bmatrix}-\begin{bmatrix}1 & -1 & 1\\2 & -2 & 2\\3 & 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1-1 & 2+1 & 3-1\\2-2 & 3+2 & 1-2\\1-3 & -1-1 & 0-3\end{bmatrix}$$

$$=\begin{bmatrix}0 & 3 & 2\\0 & 5 & -1\\-2 & -2 & -3\end{bmatrix}$$

$$C+(A-B)=\begin{bmatrix}-1 & 0 & 0\\0 & -2 & 3\\1 & 1 & 2\end{bmatrix}+\begin{bmatrix}0 & 3 & 2\\0 & 5 & -1\\-2 & -2 & -3\end{bmatrix}$$

$$=\begin{bmatrix} -1+0 & 0+3 & 0+2 \\ 0+0 & -2+5 & 3-1 \\ 1-2 & 1-2 & 2-3 \end{bmatrix}$$

$$=\begin{bmatrix} -1 & 3 & 2 \\ 0 & 3 & 2 \\ -1 & -1 & -1 \end{bmatrix}$$

L.H.S = R.H.S.

(vi) $2A+B = A+(A+B)$

L.H.S = 2A + B

$$2A+B = 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & 1 & 3 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \\ 2 & -2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & 1 & 3 \end{bmatrix}$$

$$=\begin{bmatrix} 2+1 & 4-1 & 6+1 \\ 4+2 & 6-2 & 2+2 \\ 2+3 & -2+1 & 0+3 \end{bmatrix}$$

$$=\begin{bmatrix} 3 & 3 & 7 \\ 6 & 4 & 4 \\ 5 & -1 & 3 \end{bmatrix}$$

R.H.S. = A+(A+B)

$$A+(A+B) = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix} + \left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & 1 & 3 \end{bmatrix}\right)$$

$$=\begin{bmatrix} 1 & 3 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix} + \left(\begin{bmatrix} 1+1 & 2-1 & 3+1 \\ 2+2 & 3-2 & 1+2 \\ 1+3 & -1+1 & 0+3 \end{bmatrix}\right)$$

$$=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 4 \\ 4 & 1 & 3 \\ 4 & 0 & 3 \end{bmatrix}$$

$$=\begin{bmatrix} 1+2 & 2+1 & 3+4 \\ 2+4 & 3+1 & 1+3 \\ 1+4 & -1+0 & 0+3 \end{bmatrix}$$

$$=\begin{bmatrix} 3 & 3 & 7 \\ 6 & 4 & 4 \\ 5 & -1 & 3 \end{bmatrix}$$

L.H.S. = R.H.S.

(vii) $(C-B)-A = (C-A)-B$

L.H.S. = $(C-B)-A$

$$C-B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 3 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & 1 & 3 \end{bmatrix}$$

$$=\begin{bmatrix} -1-1 & 0+1 & 0-1 \\ 0-2 & -2+2 & 3-2 \\ 1-3 & 1-1 & 2-3 \end{bmatrix}$$

$$=\begin{bmatrix} -2 & 1 & -1 \\ -2 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix}$$

$$(C-B)-A = \begin{bmatrix} -2 & 1 & -1 \\ -2 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} -2-1 & 1-2 & -1-3 \\ -2-2 & 0-3 & 1-1 \\ -2-1 & 0+1 & -1-0 \end{bmatrix}$$

$$=\begin{bmatrix} -3 & -1 & -4 \\ -4 & -3 & 0 \\ -3 & 1 & -1 \end{bmatrix}$$

R.H.S. = $(C-A)-B$

$$(C-A) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 3 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$=\begin{bmatrix} -1-1 & 0-2 & 0-3 \\ 0-2 & -2-3 & 3-1 \\ 1-1 & 1+1 & 2-0 \end{bmatrix}$$

$$=\begin{bmatrix} -2 & -2 & -3 \\ -2 & -5 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$$(C-A)-B=\begin{bmatrix}-2&-2&-3\\-2&-5&2\\0&2&2\end{bmatrix}-\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}$$

$$=\begin{bmatrix}-2-1&-2+1&-3-1\\-2-2&-5+2&2-2\\0-3&2-1&2-3\end{bmatrix}$$

$$=\begin{bmatrix}-3&-1&-4\\-4&-3&0\\-3&1&-1\end{bmatrix}$$

L.H.S = R.H.S.

(viii) $\mathbf{(A+B)+C = A+(B+C)}$

L.H.S = (A+B) + C

$$A+B=\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}+\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}$$

$$=\begin{bmatrix}1+1&2-1&3+1\\2+2&3-2&1+2\\1+3&-1+1&0+3\end{bmatrix}=\begin{bmatrix}2&1&4\\4&1&3\\4&0&3\end{bmatrix}$$

$$(A+B)+C=\begin{bmatrix}2&1&4\\4&1&3\\4&0&3\end{bmatrix}+\begin{bmatrix}-1&0&0\\0&-2&3\\1&1&2\end{bmatrix}=\begin{bmatrix}1&1&4\\4&-1&6\\5&1&5\end{bmatrix}$$

R.H.S = A + (B + C)

$$B+C=\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}+\begin{bmatrix}-1&0&0\\0&-2&3\\1&1&2\end{bmatrix}$$

$$=\begin{bmatrix}1-1&-1+0&1+0\\2+0&-2-2&2+3\\3+1&1+1&2+3\end{bmatrix}=\begin{bmatrix}0&-1&1\\2&-4&5\\4&2&5\end{bmatrix}$$

$$A+(B+C)=\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}+\begin{bmatrix}0&-1&1\\2&-4&5\\4&2&5\end{bmatrix}$$

$$=\begin{bmatrix}1+0&2-1&3+1\\2+2&3-4&1+5\\1+4&-1+5&0+5\end{bmatrix}=\begin{bmatrix}1&1&4\\4&-1&6\\5&1&5\end{bmatrix}$$

R.H.S = R.H.S

(ix) $A+(B-C)=(A-C)+B$

L.H.S = A + (B − C)

$$B-C=\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}-\begin{bmatrix}-1&0&0\\0&-2&3\\1&1&2\end{bmatrix}$$

$$=\begin{bmatrix}1+1&-1-0&1-0\\2-0&-2+2&2-3\\3-1&1-1&3-2\end{bmatrix}=\begin{bmatrix}2&-1&1\\2&0&-1\\2&0&1\end{bmatrix}$$

$$A+(B-C)=\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}+\begin{bmatrix}2&-1&1\\2&0&-1\\2&0&1\end{bmatrix}$$

$$=\begin{bmatrix}1+2&2-1&3+1\\2+2&3+0&1-1\\1+2&-1+0&0+1\end{bmatrix}$$

$$=\begin{bmatrix}3&1&4\\4&3&0\\3&-1&1\end{bmatrix}$$

R.H.S = (A−C)+B

$$A-C=\begin{bmatrix}1&2&3\\2&3&1\\1&-1&0\end{bmatrix}-\begin{bmatrix}-1&0&0\\0&-2&3\\1&1&2\end{bmatrix}$$

$$=\begin{bmatrix}1+1&2-0&3-0\\2-0&3+2&1-3\\1-1&-1-1&0-2\end{bmatrix}=\begin{bmatrix}2&2&3\\2&5&-2\\0&-2&-2\end{bmatrix}$$

$$(A-C)+B=\begin{bmatrix}2&2&3\\2&5&-2\\0&-2&-2\end{bmatrix}+\begin{bmatrix}1&-1&1\\2&-2&2\\3&1&3\end{bmatrix}$$

$$=\begin{bmatrix}2+1&2-1&3+1\\2+2&5-2&-2+2\\0+3&-2+1&-2+3\end{bmatrix}$$

$$=\begin{bmatrix}3&1&4\\4&3&0\\3&-1&1\end{bmatrix}$$

L.H.S. = R.H.S.

(x) $2A+2B=2(A+B)$

L.H.S. = **2A+2B**

$$2A+2B = 2\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \\ 2 & -2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 2 \\ 4 & -4 & 4 \\ 6 & 2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2 & 4-2 & 6+2 \\ 4+4 & 6-4 & 2+4 \\ 2+6 & -2+2 & 0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 8 \\ 8 & 2 & 6 \\ 8 & 0 & 6 \end{bmatrix}$$

R.H.S= **2 (A+B)**

$$2(A+B) = 2\left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 3 & 1 & 3 \end{bmatrix}\right)$$

$$= 2\left(\begin{bmatrix} 1+1 & 2-1 & 3+1 \\ 2+2 & 3-2 & 1+2 \\ 1+3 & -1+1 & 0+3 \end{bmatrix}\right)$$

$$= 2\begin{bmatrix} 2 & 1 & 4 \\ 4 & 1 & 3 \\ 4 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 8 \\ 8 & 2 & 6 \\ 8 & 0 & 6 \end{bmatrix}$$

L.H.S = R.H.S

6. If $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ **and** $B = \begin{bmatrix} 0 & 7 \\ -3 & 8 \end{bmatrix}$**,**

find (i) **3A–2B (ii)** $\mathbf{2A^t - 3B^t}$**.**

Ans. **(i)**

$$3A - 2B = 3\begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 0 & 7 \\ -3 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -6 \\ 9 & 12 \end{bmatrix} - \begin{bmatrix} 0 & 14 \\ -6 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 3-0 & -6-14 \\ 9+6 & 12-16 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -20 \\ 15 & -4 \end{bmatrix}$$

(ii) $\mathbf{2A^t - 3B^t}$

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$

$$2A^t = 2\begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 \\ -4 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 7 \\ -3 & 8 \end{bmatrix}$$

$$B^t = \begin{bmatrix} 0 & -3 \\ 7 & 8 \end{bmatrix}$$

$$3B^t = 3\begin{bmatrix} 0 & -3 \\ 7 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -9 \\ 21 & 24 \end{bmatrix}$$

$$2A^t - 3B^t = \begin{bmatrix} 2 & 6 \\ -4 & 8 \end{bmatrix} - \begin{bmatrix} 0 & -9 \\ 21 & 24 \end{bmatrix}$$

$$= \begin{bmatrix} 2-0 & 6+9 \\ -4-21 & 8-24 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 15 \\ -25 & -16 \end{bmatrix}$$

7. If $2\begin{bmatrix} 2 & 4 \\ -3 & a \end{bmatrix} + 3\begin{bmatrix} 1 & b \\ 8 & -4 \end{bmatrix}$

$= \begin{bmatrix} 7 & 10 \\ 18 & 1 \end{bmatrix}$**, then find a and b.**

Ans. $2\begin{bmatrix} 2 & 4 \\ -3 & a \end{bmatrix} + 3\begin{bmatrix} 1 & b \\ 8 & -4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 18 & 1 \end{bmatrix}$

$$\begin{bmatrix} 4 & 8 \\ -6 & 2a \end{bmatrix} + \begin{bmatrix} 3 & 3b \\ 24 & -12 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 18 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4+3 & 8+3b \\ -6+24 & 2a-12 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 18 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 8+3b \\ 18 & 2a-12 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 18 & 1 \end{bmatrix}$$

$\Rightarrow$ $8+3b = 10$ .... (i)

$2a - 12 = 1$ .... (ii)

From (i)

$3b = 10 - 8$

$3b = 2$

$b = \frac{2}{3}$

From (ii)

$2a = 1 + 12$

$a = \frac{13}{2}$

8. **If** $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$,

then verify that

(i) $(A+B)^t = A^t + B^t$

(ii) $(A-B)^t = A^t - B^t$

(iii) $A + A^t$ is symmetric

(iv) $A - A^t$ is skew symmetric

(v) $B + B^t$ is symmetric

(vi) $B - B^t$ is skew symmetric

Ans. **(i)** $\mathbf{(A+B)^t = A^t + B^t}$

L.H.S $= (A+B)^t$

$$(A+B) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 2+1 \\ 0+2 & 1+0 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$

$$(A+B)^t = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$$

R.H.S $= A^t + B^t$

$$A^t = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$B^t = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$A^t + B^t = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 0+2 \\ 2+1 & 1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$$

L.H.S. = R.H.S.

(ii) $(A-B)^t = A^t - B^t$

L.H.S. $= (A-B)^t$

$$(A-B) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$$

$$(A-B) = \begin{bmatrix} 1-1 & 2-1 \\ 0-2 & 1-0 \end{bmatrix}$$

$$(A-B) = \begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$$

$$(A-B)^t = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

R.H.S $= A^t - B^t$

$$A^t = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$B^t = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$A^t - B^t = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 0-2 \\ 2-1 & 1-0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

L.H.S = R.H.S

(iii) $A + A^t$ **is symmetric**

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$A + A^t = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 2+0 \\ 0+2 & 1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$(A + A^t)^t = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = A + A^t$$

So, $A + A^t$ is symmetric.

(iv) **$A - A^t$ is skew symmetric**

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$A^t = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$A - A^t = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & 2-0 \\ 0-2 & 1-1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$\left(A - A^t\right)^t = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$$

$$\left(A - A^t\right)^t = -\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$= -\left(A - A^t\right)^t$ is skew symmetric

(v) **$B+B^t$ is symmetric**

$$B + B^t = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1+1 & 1+2 \\ 2+1 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 3 & 0 \end{bmatrix}$$

$$\left(B + B^t\right)^t = \begin{bmatrix} 2 & 3 \\ 3 & 0 \end{bmatrix}$$

$= (B+B^t)$ is symmetric

(vi) $B-B^t$ is skew symmetric

$$B - B^t = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1-1 & 1-2 \\ 2-1 & 0-0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\left(B - B^t\right)^t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$= -\left(B - B^t\right)$ is skew symmetric

Multiplication of Matrices.

Two matrices A and B are conformable for multiplication, giving product AB if the number of columns of A is equal to the number of rows of B.

e.g., let $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$. Here number of columns of A is equal to the number of rows of B. So A and B matrices are conformable for multiplication.

Examples

(i) **If** $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ **and** $B = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$,

then $AB = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1\times2+2\times3 & 1\times0+2\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+6 & 0+2 \end{bmatrix} = \begin{bmatrix} 8 & 2 \end{bmatrix}$$

It is a matrix of order 1-by-2.

(ii)

If $A = \begin{bmatrix} 1 & 3 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$, then

$$AB=\begin{bmatrix}1 & 3\\ 2 & -3\end{bmatrix}\times\begin{bmatrix}-1 & 0\\ 3 & 2\end{bmatrix}$$

$$=\begin{bmatrix}1\times(-1)+3\times3 & 1\times0+3\times2\\ 2(-1)+(-3)(3) & 2\times0+(-3)(2)\end{bmatrix}$$

$$=\begin{bmatrix}-1+9 & 0+6\\ -2-9 & 0-6\end{bmatrix}=\begin{bmatrix}8 & 6\\ -11 & -6\end{bmatrix},$$ is a 2-by-2 matrix.

Associative Law under Multiplication

If A, B and C are three matrices conformable for multiplication then associative law under multiplication is given as

(AB)C=A(BC)

e.g., If $A=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}$, $B=\begin{bmatrix}0 & 1\\ 3 & 1\end{bmatrix}$ and

$C=\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}$, then

L.H.S. = (AB)C

$$=\left(\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 3 & 1\end{bmatrix}\right)\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}2\times0+3\times3 & 2\times1+3\times1\\ -1\times0+0\times3 & -1\times1+0\times1\end{bmatrix}\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}0+9 & 2+3\\ 0+0 & -1+0\end{bmatrix}\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}9 & 5\\ 0 & -1\end{bmatrix}\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}9\times2+5\times(-1) & 9\times2+5\times0\\ 0\times2+(-1)\times(-1) & 0\times2+(-1)\times0\end{bmatrix}$$

$$=\begin{bmatrix}18-5 & 18+0\\ 0+1 & 0+0\end{bmatrix}=\begin{bmatrix}13 & 18\\ 1 & 0\end{bmatrix}$$

$$R.H.S=A(BC)=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\left(\begin{bmatrix}0 & 1\\ 3 & 1\end{bmatrix}\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}\right)$$

$$=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\begin{bmatrix}0\times2+1\times(-1) & 0\times2+1\times0\\ 3\times2+1\times(-1) & 3\times2+1\times0\end{bmatrix}$$

$$=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\begin{bmatrix}-1 & 0\\ 5 & 6\end{bmatrix}$$

$$=\begin{bmatrix}2(-1)+3\times5 & 2\times0+3\times6\\ (-1)(-1)+0\times5 & -1\times0+0\times6\end{bmatrix}$$

$$=\begin{bmatrix}-2+15 & 0+18\\ 1+0 & 0+0\end{bmatrix}$$

$$=\begin{bmatrix}13 & 18\\ 1 & 0\end{bmatrix}=(AB)C$$

The associative law under multiplication of matrices is verified.

Distributive Laws of Multiplication over Addition and Subtraction

(a) Let A, B and C be three matrices. Then distributive laws of multiplication over addition are given below.

(i) $A(B+C)=AB+AC$

(Left distributive law)

(ii) $(A+B)C=AC+BC$

(Right distributive law)

Let $A=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}$, $B=\begin{bmatrix}0 & 1\\ 3 & 1\end{bmatrix}$

and $C=\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}$ then in (i)

L.H.S. = A(B+C)

$$=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\left(\begin{bmatrix}0 & 1\\ 3 & 1\end{bmatrix}+\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}\right)$$

$$=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\begin{bmatrix}0+2 & 1+2\\ 3-1 & 1+0\end{bmatrix}$$

$$=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\begin{bmatrix}2 & 3\\ 2 & 1\end{bmatrix}$$

$$=\begin{bmatrix}2\times2+3\times2 & 2\times3+3\times1\\ -1\times2+0\times2 & -1\times3+0\times1\end{bmatrix}$$

$$=\begin{bmatrix}4+6 & 6+3\\ -2+0 & -3+0\end{bmatrix}=\begin{bmatrix}10 & 9\\ -2 & -3\end{bmatrix}$$

R.H.S. = AB + AC

$$=\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 3 & 1\end{bmatrix}+\begin{bmatrix}2 & 3\\ -1 & 0\end{bmatrix}\begin{bmatrix}2 & 2\\ -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}2\times0+3\times3 & 2\times1+3\times1\\ -1\times0+0\times3 & -1\times1+0\times1\end{bmatrix}$$

$$+\begin{bmatrix}2\times2+3\times(-1) & 2\times2+3\times0\\ -1\times2+0\times(-1) & -1\times2+0\times0\end{bmatrix}$$

$$=\begin{bmatrix}9 & 5\\ 0 & -1\end{bmatrix}+\begin{bmatrix}1 & 4\\ -2 & -2\end{bmatrix}$$

$$=\begin{bmatrix}9+1 & 5+4\\ 0-2 & -1-2\end{bmatrix}=\begin{bmatrix}10 & 9\\ -2 & -3\end{bmatrix}=\text{L.H.S}$$

Which shows that

$$A(B+C)=AB+AC;$$

b) **Let** $A=\begin{bmatrix}2 & 3\\ 0 & 1\end{bmatrix}$, $B=\begin{bmatrix}-1 & 1\\ 1 & 0\end{bmatrix}$

and $C=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$, then in (i)

L.H.S. $=A(B-C)$

$$=\begin{bmatrix}2 & 3\\ 0 & 1\end{bmatrix}\left(\begin{bmatrix}-1 & 1\\ 1 & 0\end{bmatrix}-\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)$$

$$=\begin{bmatrix}2 & 3\\ 0 & 1\end{bmatrix}\left(\begin{bmatrix}-1-2 & 1-1\\ 1-1 & 0-2\end{bmatrix}\right)$$

$$=\begin{bmatrix}2 & 3\\ 0 & 1\end{bmatrix}\begin{bmatrix}-3 & 0\\ 0 & -2\end{bmatrix}$$

$$=\begin{bmatrix}(2)(-3)+(3)(0) & 2(0)+3(-2)\\ (0)(-3)+1\times0 & 0\times0+(1)(-2)\end{bmatrix}$$

$$=\begin{bmatrix}-6+0 & 0-6\\ 0+0 & 0-2\end{bmatrix}=\begin{bmatrix}-6 & -6\\ 0 & -2\end{bmatrix}$$

R.H.S. = AB − AC

$$=\begin{bmatrix}2 & 3\\ 0 & 1\end{bmatrix}\begin{bmatrix}-1 & 1\\ 1 & 0\end{bmatrix}-\begin{bmatrix}2 & 3\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}2(-1)+3(1) & 2(1)+3(0)\\ 0(-1)+1(1) & 0(1)+1(0)\end{bmatrix}$$

$$-\begin{bmatrix}2\times2+3\times1 & 2\times1+3\times2\\ 0\times2+1\times1 & 0\times1+1\times2\end{bmatrix}$$

$$=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix}-\begin{bmatrix}7 & 8\\ 1 & 2\end{bmatrix}=\begin{bmatrix}1-7 & 2-8\\ 1-1 & 0-2\end{bmatrix}$$

$$=\begin{bmatrix}-6 & -6\\ 0 & -2\end{bmatrix}$$

Which shows that

$$A(B-C)=AB-AC$$

Commutative Law of Multiplication of Matrices

Consider the matrices $A=\begin{bmatrix}0 & 1\\ 2 & 3\end{bmatrix}$ and

$B=\begin{bmatrix}1 & 1\\ 0 & -2\end{bmatrix}$ then

$$AB=\begin{bmatrix}0 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}$$

$$=\begin{bmatrix}0\times1+1\times0 & 0\times0+1(-2)\\ 2\times1+3\times0 & 2\times0+3(-2)\end{bmatrix}$$

$$=\begin{bmatrix}0 & -2\\ 2 & -6\end{bmatrix}$$

and

$$BA=\begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}0 & 1\\ 2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1\times0+0\times2 & 1\times1+0\times3\\ 0\times0+(-2)\times2 & 0\times1+3(-2)\end{bmatrix}$$

$$=\begin{bmatrix} 0 & 1 \\ -4 & -6 \end{bmatrix}$$

Which shows that. $AB \neq BA$.

Note: Commutative law under multiplication in matrices does not hold in general i.e., if A and B are two matrices then $AB \neq BA$.

Commutative law under multiplication holds in particular case.

e.g., If $A=\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}$

then

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}$$

$$=\begin{bmatrix} 2\times(-3)+0\times0 & 2\times0+0\times4 \\ 0\times(-3)+1\times0 & 0\times0+1\times4 \end{bmatrix}$$

$$=\begin{bmatrix} -6 & 0 \\ 0 & 4 \end{bmatrix}$$

and $$BA = \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} -3\times2+0\times0 & -3\times0+0\times1 \\ 0\times2+4\times0 & 0\times0+4\times1 \end{bmatrix}$$

$$=\begin{bmatrix} -6 & 0 \\ 0 & 4 \end{bmatrix}$$

Which shows that $AB = BA$.

Multiplicative Identity of a Matrix.

Let A be a matrix. Another matrix B is called the identity matrix of A under multiplication if

$AB = A = BA$

If $A=\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then

$$AB=\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1\times1+2\times0 & 1\times0+2\times1 \\ 0\times1+(-3)\times0 & 0\times0+(-3)(1) \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$$

$$BA=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$$

$$=\begin{bmatrix} 1\times1+0\times0 & 1\times2+0\times(-3) \\ 0\times1+1\times0 & 0\times2+1\times(-3) \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$$

Which shows that $AB = A = BA$.

Verification of $(AB)^t = B^t A^t$.

If A and B are two matrices and A^t, B^t are their respective transpose,

then $(AB)^t = B^t A^t$.

e.g., $A=\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}$

L.H.S. $= (AB)^t$

$$=\left(\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}\right)^t$$

$$=\begin{bmatrix} 2\times1+1\times(-2) & 2\times3+1\times0 \\ 0\times1+(-1)\times(-2) & 0\times3+(-1)\times0 \end{bmatrix}^t$$

$$=\begin{bmatrix} 2-2 & 6+0 \\ 0+2 & 0+0 \end{bmatrix}^t=\begin{bmatrix} 0 & 6 \\ 2 & 0 \end{bmatrix}^t$$

$$=\begin{bmatrix} 0 & 2 \\ 6 & 0 \end{bmatrix}$$

R.H.S. $= B^t A^t$,

$$(A)^t = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}^t = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

$$(B)^t = \begin{bmatrix} 1 & 3 \\ -2 & 0 \end{bmatrix}^t = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$$

$$B^t A^t = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times2+(-2)\times1 & 1\times0+(-2)(-1) \\ 3\times2+0\times1 & 3\times0+0\times(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 2-2 & 0+2 \\ 6+0 & 0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 \\ 6 & 0 \end{bmatrix} = \text{L.H.S}$$

L.H.S = R.H.S

Thus $(AB)^t = B^t A^t$.

Exercise 1.4

1. Which of the following product of matrices is conformable for multiplication?

Ans. (i) $\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

Number of Columns = Number of Rows

$\therefore$ product is possible.

(ii) $\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

Number of columns = Number of Rows.

$\therefore$ product is possible.

(iii) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$

Number of columns $\neq$ Number of Rows.

$\therefore$ product is not possible.

(iv) $\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$

Number of columns = Number of Rows.

$\therefore$ product is possible.

(v) $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ -2 & 3 \end{bmatrix}$

Number of Columns = Number of Rows.

$\therefore$ Product is possible.

2. If $A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$, find (i) AB (ii) BA (if possible).

(i) $$AB = \begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3(6)+0(5) \\ -1(6)+2(5) \end{bmatrix}$$

$$= \begin{bmatrix} 18+0 \\ -6+10 \end{bmatrix} = \begin{bmatrix} 18 \\ 4 \end{bmatrix}$$

(ii) $$BA = \begin{bmatrix} 6 \\ 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$$

$\therefore$ Product is not possible.

Because number of columns $\neq$ number of rows.

3. Find the following products.

Ans. (i) $\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}4\\0\end{bmatrix}$

$=\begin{bmatrix}1(4)+2(0)\end{bmatrix}$

$=\begin{bmatrix}4+0\end{bmatrix}$

$=\begin{bmatrix}4\end{bmatrix}$

(ii) $\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}5\\-4\end{bmatrix}$

$=\begin{bmatrix}1(5)+2(-4)\end{bmatrix}$

$=\begin{bmatrix}5-8\end{bmatrix}$

$=\begin{bmatrix}-3\end{bmatrix}$

(iii) $\begin{bmatrix}-3 & 0\end{bmatrix}\begin{bmatrix}4\\0\end{bmatrix}$

$=\begin{bmatrix}-3(4)+0(0)\end{bmatrix}=\begin{bmatrix}-12\end{bmatrix}$

(iv) $\begin{bmatrix}6 & 0\end{bmatrix}\begin{bmatrix}4\\0\end{bmatrix}$

$=\begin{bmatrix}6(4)+(0)(0)\end{bmatrix}=\begin{bmatrix}24\end{bmatrix}$

(v) $\begin{bmatrix}1 & 2\\-3 & 0\\6 & -1\end{bmatrix}\begin{bmatrix}4 & 5\\0 & -4\end{bmatrix}$

$=\begin{bmatrix}1(4)+2(0) & 1(5)+2(-4)\\-3(4)+0(0) & -3(5)+0(-4)\\6(4)+-1(0) & 6(5)+(-1)(-4)\end{bmatrix}$

$=\begin{bmatrix}4+0 & 5-8\\-12+0 & -15+0\\24+0 & 30+4\end{bmatrix}$

$=\begin{bmatrix}4 & -3\\-12 & -15\\24 & 34\end{bmatrix}$

4. Multiply the following matric

(a) $\begin{bmatrix}2 & 3\\1 & 1\\0 & -2\end{bmatrix}\begin{bmatrix}2 & -1\\3 & 0\end{bmatrix}$

(b) $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\\-1 & 1\end{bmatrix}$

(c) $\begin{bmatrix}1 & 2\\3 & 4\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$

(d) $\begin{bmatrix}8 & 5\\6 & 4\end{bmatrix}\begin{bmatrix}2 & -\frac{5}{4}\\4 & 4\end{bmatrix}$

(e) $\begin{bmatrix}-1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

Ans. (a) $\begin{bmatrix}2 & 3\\1 & 1\\0 & -2\end{bmatrix}\begin{bmatrix}2 & -1\\3 & 0\end{bmatrix}$

$=\begin{bmatrix}2(2)+3(3) & 2(-1)+3(0)\\1(2)+1(3) & 1(-1)+1(0)\\0(2)+(-2)(3) & 0(-1)+(-2)(0)\end{bmatrix}$

$=\begin{bmatrix}13 & -2\\5 & -1\\-6 & 0\end{bmatrix}$

(b) $\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}\begin{bmatrix}1 & 2\\3 & 4\\-1 & 1\end{bmatrix}$

$=\begin{bmatrix}1(1)+2(3)+3(-1) & 1(2)+2(4)+3(1)\\4(1)+15(3)+6(-1) & 4(2)+5(4)+6(1)\end{bmatrix}$

$=\begin{bmatrix}1+6-3 & 2+8+3\\4+15-6 & 8+20+6\end{bmatrix}$

$$=\begin{bmatrix}4 & 13\\13 & 34\end{bmatrix}$$

(c) $\begin{bmatrix}1 & 2\\3 & 4\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$

$$=\begin{bmatrix}1(1)+2(4) & 1(2)+2(5) & 1(3)+2(6)\\3(1)+4(4) & 3(2)+4(5) & 3(3)+4(6)\\-1(1)+1(4) & -1(2)+1(5) & -1(3)+1(6)\end{bmatrix}$$

$$=\begin{bmatrix}9 & 12 & 15\\19 & 26 & 33\\3 & 3 & 3\end{bmatrix}$$

(d) $\begin{bmatrix}8 & 5\\6 & 4\end{bmatrix}\begin{bmatrix}2 & -\frac{5}{2}\\-4 & 4\end{bmatrix}$

$$=\begin{bmatrix}8(2)+5(-4) & 8\left(\frac{-5}{2}\right)+5(4)\\6(2)+4(-4) & 6\left(\frac{-5}{2}\right)+4(4)\end{bmatrix}$$

$$=\begin{bmatrix}16-20 & -20+20\\12-16 & -15+16\end{bmatrix}=\begin{bmatrix}-4 & 0\\-4 & 1\end{bmatrix}$$

(e) $\begin{bmatrix}-1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$

$$=\begin{bmatrix}-1(0)+2(0) & -1(0)+2(0)\\1(0)+3(0) & 1(0)+3(0)\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

5. Let $A=\begin{bmatrix}-1 & 3\\2 & 0\end{bmatrix}$, $B=\begin{bmatrix}1 & 2\\-3 & -5\end{bmatrix}$ **and** $C=\begin{bmatrix}2 & 1\\1 & 3\end{bmatrix}$. **Verify whether**

(i) AB = BA.

(ii) A(BC) = (AB)C

(iii) A(B+C)=AB+AC

(iv) A(B−C)=AB−AC

Ans. (i) AB = BA.

To check whether AB = BA Or not

$$AB=\begin{bmatrix}-1 & 3\\2 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\-3 & -5\end{bmatrix}$$

$$=\begin{bmatrix}-1(1)+3(-3) & -1(2)+3(-5)\\2(1)+0(-3) & 2(2)+0(-5)\end{bmatrix}$$

$$=\begin{bmatrix}-1-9 & -2-15\\2-0 & 4+0\end{bmatrix}$$

$$=\begin{bmatrix}-10 & -17\\2 & 4\end{bmatrix}$$

$$BA=\begin{bmatrix}1 & 2\\-3 & -5\end{bmatrix}\begin{bmatrix}-1 & 3\\2 & 0\end{bmatrix}$$

$$=\begin{bmatrix}1(-1)+2(2) & 1(3)+2(0)\\-3(-1)+-5(2) & -3(3)+(-5)(0)\end{bmatrix}$$

$$=\begin{bmatrix}-1+4 & 3+0\\3-10 & -9+0\end{bmatrix}$$

$$=\begin{bmatrix}3 & 3\\-7 & -9\end{bmatrix}$$

So **AB ≠ BA**

(ii) A(BC) = (AB)C

L.H.S = A(BC)

$$BC=\begin{bmatrix}1 & 2\\-3 & -5\end{bmatrix}\begin{bmatrix}2 & 1\\1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1(2)+2(1) & 1(1)+2(3)\\-3(2)+-5(1) & -3(1)+-5(3)\end{bmatrix}$$

$$=\begin{bmatrix}2+2 & 1+6\\-6-5 & -3-15\end{bmatrix}$$

$$=\begin{bmatrix}4 & 7\\-11 & -18\end{bmatrix}$$

$$A(BC)=\begin{bmatrix}-1 & 3\\2 & 0\end{bmatrix}\begin{bmatrix}4 & 7\\-11 & -18\end{bmatrix}$$

$$=\begin{bmatrix}-1(4)+3(-11) & -1(7)+3(-18)\\2(4)+0(-11) & 2(7)+0(-18)\end{bmatrix}$$

$$=\begin{bmatrix}-4-33 & -7-54\\8+0 & 14+0\end{bmatrix}$$

$$=\begin{bmatrix}-37 & -61\\ 8 & 14\end{bmatrix}$$

R.H.S = (AB)C

$$(AB)=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}$$

$$=\begin{bmatrix}-1(1)+3(-3) & -1(2)+3(-5)\\ 2(1)+0(-3) & 2(2)+0(-5)\end{bmatrix}$$

$$=\begin{bmatrix}-1-9 & -2-15\\ 2+0 & 4+0\end{bmatrix}=\begin{bmatrix}-10 & -17\\ 2 & 4\end{bmatrix}$$

$$(AB)C=\begin{bmatrix}-10 & -17\\ 2 & 4\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-10(2)+(-17)(1) & -10(1)+(-17)(3)\\ 2(2)+4(1) & 2(1)+4(3)\end{bmatrix}$$

$$=\begin{bmatrix}-20-17 & -10-51\\ 4+4 & 2+12\end{bmatrix}$$

$$=\begin{bmatrix}-37 & -61\\ 8 & 14\end{bmatrix}$$

Hence A(BC) = (AB)C

(iii) A (B + C) = AB + AC

L.H.S = A (B + C)

$$(B+C)=\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}+\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1+2 & 2+1\\ -3+1 & -5+3\end{bmatrix}=\begin{bmatrix}3 & 3\\ -2 & -2\end{bmatrix}$$

$$A(B+C)=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}3 & 3\\ -2 & -2\end{bmatrix}=\begin{bmatrix}-3-6 & -3-6\\ 6+0 & 6+0\end{bmatrix}$$

$$=\begin{bmatrix}-9 & -9\\ 6 & 6\end{bmatrix}$$

L.H.S.

AB + AC

$$AB=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}$$

$$=\begin{bmatrix}-1(1)+3(-3) & -1(2)+3(-5)\\ 2(1)+0(-3) & 2(2)+0(-5)\end{bmatrix}$$

$$=\begin{bmatrix}-1-9 & -2-15\\ 2+0 & 4+0\end{bmatrix}=\begin{bmatrix}-10 & -17\\ 2 & 4\end{bmatrix}$$

$$AC=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-1(2)+3(1) & -1(1)+3(3)\\ 2(2)+0(1) & 2(1)+0(3)\end{bmatrix}=\begin{bmatrix}1 & 8\\ 4 & 2\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}-10 & -17\\ 2 & 4\end{bmatrix}+\begin{bmatrix}1 & 8\\ 4 & 2\end{bmatrix}$$

$$=\begin{bmatrix}-10+1 & -17+8\\ 2+4 & 4+2\end{bmatrix}=\begin{bmatrix}-9 & -9\\ 6 & 6\end{bmatrix}$$

6. For the matrices.

$$A=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix},\ B=\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix},\ C=\begin{bmatrix}-2 & 6\\ 3 & -9\end{bmatrix}$$

Verify that(i)$(AB)^t = B^t A^t$(ii)$(BC)^t = C^t B^t$

Ans. (i) $(AB)^t = B^t A^t$

L.H.S = $(AB)^t$

$$AB=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}$$

$$=\begin{bmatrix}-1(1)+3(-3) & -1(2)+3(-5)\\ 2(1)+0(-3) & 2(2)+0(-5)\end{bmatrix}$$

$$=\begin{bmatrix}-1-9 & -2-15\\ 2+0 & 4+0\end{bmatrix}$$

$$=\begin{bmatrix}-10 & -17\\ 2 & 4\end{bmatrix}$$

$$(AB)^t=\begin{bmatrix}-10 & 2\\ -17 & 4\end{bmatrix}$$

R.H.S = $B^t A^t$

$$A^t=\begin{bmatrix}-1 & 2\\ 3 & 0\end{bmatrix}$$

$$B^t=\begin{bmatrix}1 & -3\\ 2 & -5\end{bmatrix}$$

$$B^tA^t=\begin{bmatrix}1 & -3\\ 2 & -5\end{bmatrix}\begin{bmatrix}-1 & 2\\ 3 & 0\end{bmatrix}$$

$$=\begin{bmatrix}1(-1)+(-3)(3) & 1(2)+(-3)(0)\\ 2(-1)+(-5)(3) & 2(2)+5(0)\end{bmatrix}$$

$$=\begin{bmatrix}-1-9 & 2-0\\ -2-15 & 4\end{bmatrix}=\begin{bmatrix}-10 & 2\\ -17 & 4\end{bmatrix}$$

L.H.S = R.H.S

Hence $(AB)^t = B^t A^t$

(ii) $(BC)^t = C^t B^t$

L.H.S $= (BC)^t$

$$BC = \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix}\begin{bmatrix} -2 & 6 \\ 3 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 1(-2)+2(3) & 1(6)+2(-9) \\ -3(-2)+-5(3) & -3(6)+-5(-9) \end{bmatrix}$$

$$= \begin{bmatrix} -2+6 & 6-18 \\ 6-15 & -18+45 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -12 \\ -9 & 27 \end{bmatrix}$$

$$(BC)^t = \begin{bmatrix} 4 & -9 \\ -12 & 27 \end{bmatrix}$$

R.H.S $= C^t B^t$

$$C^t = \begin{bmatrix} -2 & 3 \\ 6 & -9 \end{bmatrix}$$

$$B^t = \begin{bmatrix} 1 & -3 \\ 2 & -5 \end{bmatrix}$$

$$C^tB^t = \begin{bmatrix} -2 & 3 \\ 6 & -9 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 2 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} -2(1)+3(2) & -2(-3)+3(-5) \\ 6(1)+2(-9) & 6(-3)+-9(-5) \end{bmatrix}$$

$$= \begin{bmatrix} -2+6 & 6-15 \\ 6-18 & -18+45 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -9 \\ -12 & 27 \end{bmatrix}$$

L.H.S = R.H.S

Hence $(BC)^t = C^t B^t$

Determinant of a 2-by-2 Matrix.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2-by-2 square matrix. The determinant of A, denoted by det A or |A| is defined as $|A| = \det A = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \lambda \in R$ e.g.,

Let $B = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$. Then $|B| = \det B = \begin{vmatrix} 1 & 1 \\ -2 & 3 \end{vmatrix}$

$$= 1\times 3 - (-2)(1) = 3 + 2 = 5$$

If $M = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$, then

$$\det M = \begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix} = 2\times 3 - 1\times 6 = 0$$

Singular and non-singular matrix.

A square matrix A is called singular if determinant of A is equal to zero.

i.e., $|A| = 0$.

For example, $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ is a singular matrix, since $\det A = 1\times 0 - 0\times 2 = 0$

A square matrix A is called non-singular if the determinant of A is not equal to zero.

i.e., $|A| \neq 0$

For example $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ is non-singular, since $\det A = 1\times 2 - 0\times 1 = 2 \neq 0$.

Note that, each square matrix with real entries is either singular or non-singular.

Adjoint of a Matrix.

Adjoint of a square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is obtained by interchanging the diagonal entries and changing the sign of other entries. Adjoint of matrix A is denoted as Adj A.

$$\text{i.e., Adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

e.g., if $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$, then

$$\text{Adj } A = \begin{bmatrix} 0 & -2 \\ -3 & 1 \end{bmatrix}$$

If $B = \begin{bmatrix} 2 & -1 \\ 3 & -4 \end{bmatrix}$, then $\text{Adj } B = \begin{bmatrix} -4 & 1 \\ -3 & 2 \end{bmatrix}$

Multiplicative inverse of a non-singular matrix.

Let A and B be two non-singular square matrices of same order. Then A and B are said to be multiplicative inverse of each other if

$$AB = BA = I$$

The inverse of A is denoted by A^{-1}, **thus $AA^{-1} = A^{-1}A = I$.**

Inverse of a matrix is possible only if matrix is non-singular.

Inverse of a Matrix using Adjoint

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a square matrix. To find the inverse of M, i.e., M^{-1}, first we find the determinant as inverse is possible only of a non-singular matrix.

$$|M| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$$

and $\text{Adj } M = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, then

$$M^{-1} = \frac{\text{Adj } M}{|M|}$$

e.g., Let $A = \begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix}$.

Then $|A| = -6 - (-1) = 6 + 1 = -5 \neq 0$

$$|A| = -6 - (-1) = -6 + 1 = -5 \neq 0.$$

Thus $A^{-1} = \dfrac{\text{Adj } A}{|A|} = \dfrac{\begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix}}{-5}$

$$= \frac{-1}{5}\begin{bmatrix} -3 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-1}{5} & \frac{-2}{5} \end{bmatrix}$$

and $AA^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{-1}{5} & \frac{-2}{5} \end{bmatrix}$

$$= \begin{bmatrix} \frac{6}{5} - \frac{1}{5} & \frac{2}{5} - \frac{2}{5} \\ -\frac{3}{5} + \frac{3}{5} & -\frac{1}{5} + \frac{6}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = A^{-1}A$$

Verification of $(AB)^{-1} = B^{-1}A^{-1}$

Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}$

Then $\det A = 3 \times 0 - (-1) \times 1 = 1 \neq 0$

And $\det B = 0 \times 2 - 3(-1) = 3 \neq 0$

Therefore, A and B are invertible i.e., their inverses exist.

Then, to verify the law of inverse of the product, take

AB

$$= \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times0+1\times3 & 3\times(-1)+1\times2 \\ -1\times0+0\times3 & -1\times(-1)+0\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \det(AB) = \begin{vmatrix} 3 & -1 \\ 0 & 1 \end{vmatrix} = 3 \neq 0$$

and L.H.S. = $(AB)^{-1}$

$$(AB)^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & 1 \end{bmatrix}$$

R.H.S.= $B^{-1}A^{-1}$, where $B^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}$,

$$A^{-1} = \frac{1}{1}\begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -3 & 0 \end{bmatrix}\frac{1}{1}\begin{bmatrix} 0 & -1 \\ 1 & 3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2\times 0 + 1\times 1 & 2\times(-1) + 1\times 3 \\ -3\times 0 + 0\times 1 & -3\times(-1) + 0\times 3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 0+1 & -2+3 \\ 0 & 3 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & 1 \end{bmatrix} = (AB)^{-1}$$

Thus the law $(AB)^{-1} = B^{-1}A^{-1}$ is verified.

Exercise 1.5

1. Find the determinant of the following matrices.

Ans. (i) $A = \begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix}$$
$$= -1(0) - 2(1)$$
$$= 0 - 2 = -2$$

(ii) $B = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}$

$$|B| = 1(-2) - 2(3)$$
$$= -2 - 6$$
$$= -8$$

(iii) $C = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}$

$$|C| = 3(2) - 3(2)$$
$$= 6 - 6 = 0$$

(iv) $D = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

$$|D| = 3(4) - 1(2)$$
$$= 12 - 2 = 10$$

2. Find which of the following matrices are singular or non-singular?

Ans. (i) $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 6 \\ 2 & 4 \end{vmatrix}$$
$$= 3(4) - 2(6)$$
$$= 12 - 12$$
$$= 0 \quad \text{singular}$$

(ii) $B = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$

$$|B| = \begin{vmatrix} 4 & 1 \\ 3 & 2 \end{vmatrix}$$
$$= 4(2) - 3(1) = 8 - 3 = 5 \quad \text{non-singular}$$

(iii) $C = \begin{bmatrix} 7 & -9 \\ 3 & 5 \end{bmatrix}$

$$|C| = \begin{vmatrix} 7 & -9 \\ 3 & 5 \end{vmatrix}$$
$$= 7(5) - 3(-9)$$
$$= 35 + 27$$
$$= 62 \neq 0 \quad \text{non-singular}$$

(iv) $D=\begin{bmatrix}5 & -10\\ -2 & 4\end{bmatrix}$

$|D|=\begin{vmatrix}5 & -10\\ -2 & 4\end{vmatrix}$

$=5(4)-(-2)(-10)$

$=20-20$

$=0$ singular

3. Find the multiplicative inverse (if it exists) of each.

Ans. (i) $A=\begin{bmatrix}-1 & 3\\ 2 & 0\end{bmatrix}$

$|A|=\begin{vmatrix}-1 & 3\\ 2 & 0\end{vmatrix}$

$=-1(0)-2(3)$

$=-6$

$\text{Adj}A=\begin{bmatrix}0 & -3\\ -2 & -1\end{bmatrix}$

$A^{-1}=\frac{1}{|A|}\text{adj }A$

$=\frac{1}{-6}\begin{bmatrix}0 & -3\\ -2 & -1\end{bmatrix}=\frac{1}{6}\begin{bmatrix}0 & 3\\ 2 & 1\end{bmatrix}$

$=\begin{bmatrix}0 & \frac{1}{2}\\ \frac{1}{3} & \frac{1}{6}\end{bmatrix}$

(ii) $B=\begin{bmatrix}1 & 2\\ -3 & -5\end{bmatrix}$

$|B|=\begin{vmatrix}1 & 2\\ -3 & -5\end{vmatrix}$

$=1(-5)-(-3)(2)$

$=-5+6$

$=1\neq 0$

$\text{Adj }B=\begin{bmatrix}-5 & -2\\ 3 & 1\end{bmatrix}$

$B^{-1}=\frac{1}{|B|}\text{adj}B$

$=\frac{1}{1}\begin{bmatrix}-5 & -2\\ 3 & 1\end{bmatrix}=\begin{bmatrix}-5 & -2\\ 3 & 1\end{bmatrix}$

(iii) $C=\begin{bmatrix}-2 & 6\\ 3 & -9\end{bmatrix}$

$|C|=\begin{vmatrix}-2 & 6\\ 3 & -9\end{vmatrix}$

$=-2(-9)-3(6)$

$=18-18=0$

C^{-1} does not exist.

(iv) $D=\begin{bmatrix}\frac{1}{2} & \frac{3}{4}\\ 1 & 2\end{bmatrix}$

$|D|=\begin{vmatrix}\frac{1}{2} & \frac{3}{4}\\ 1 & 2\end{vmatrix}$

$=\frac{1}{2}(2)-1\left(\frac{3}{4}\right)$

$=1-\frac{3}{4}$

$=\frac{4-3}{4}=\frac{1}{4}\neq 0$

$\text{Adj }D=\begin{bmatrix}2 & \frac{-3}{4}\\ -1 & \frac{1}{2}\end{bmatrix}$

$D^{-1}=\frac{1}{|D|}\text{adj.}D$

$=\frac{1}{1/4}\begin{bmatrix}2 & -3/4\\ -1 & 1/2\end{bmatrix}$

$=4\begin{bmatrix}2 & \frac{-3}{4}\\ -1 & \frac{1}{2}\end{bmatrix}=\begin{bmatrix}8 & -3\\ -4 & 2\end{bmatrix}$

4. If $A=\begin{bmatrix}1 & 2\\4 & 6\end{bmatrix}$ and $B=\begin{bmatrix}3 & -1\\2 & -2\end{bmatrix}$, then

(i) $A(\text{Adj }A) = (\text{Adj }A)\,A = (\det A)I$

(ii) $BB^{-1} = I = B^{-1}B$

Ans. (i) $A(\text{Adj }A)=(\text{Adj }A)\,A = (\det A)I$

$$A=\begin{bmatrix}1 & 2\\4 & 6\end{bmatrix}$$

$$\text{Adj}A=\begin{bmatrix}6 & -2\\-4 & 1\end{bmatrix}$$

$$A(\text{Adj}A)=\begin{bmatrix}1 & 2\\4 & 6\end{bmatrix}\begin{bmatrix}6 & -2\\-4 & 1\end{bmatrix}$$

$$=\begin{bmatrix}1(6)+2(-4) & 1(-2)+2(1)\\4(6)+6(-4) & 4(-2)+6(1)\end{bmatrix}$$

$$=\begin{bmatrix}6-8 & -2+2\\24-24 & -8+6\end{bmatrix}$$

$$=\begin{bmatrix}-2 & 0\\0 & -2\end{bmatrix}$$

$$\text{Now } (\text{Adj}A)A=\begin{bmatrix}6 & -2\\-4 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\4 & 6\end{bmatrix}$$

$$=\begin{bmatrix}6(1)+-2(4) & 6(2)+-2(6)\\-4(1)+1(4) & -4(2)+1(6)\end{bmatrix}$$

$$=\begin{bmatrix}6-8 & 12-12\\-4+4 & -8+6\end{bmatrix}$$

$$=\begin{bmatrix}-2 & 0\\0 & -2\end{bmatrix}$$

Also $(\det A)I$

$$\det A =\begin{vmatrix}1 & 2\\4 & 6\end{vmatrix}$$

$$=1(6)-2(4)=6-8=-2$$

$$(\det A)I=-2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}-2 & 0\\0 & -2\end{bmatrix}$$

Hence: $A(\text{Adj}A) = (\text{Adj}A)\,A = (\det A)I$

(ii) $B=\begin{bmatrix}3 & -1\\2 & -2\end{bmatrix}$

$$|B|=\begin{vmatrix}3 & -1\\2 & -2\end{vmatrix}=3(2)-2(-1)$$

$$=-6+2=-4\neq 0$$

$$\text{Adj}B=\begin{bmatrix}-2 & 1\\-2 & 3\end{bmatrix}$$

$$B^{-1}=\frac{1}{|B|}\text{Adj}B$$

$$=\frac{1}{-4}\begin{bmatrix}-2 & 1\\-2 & 3\end{bmatrix}=\frac{1}{4}\begin{bmatrix}2 & -1\\2 & -3\end{bmatrix}$$

$$BB^{-1}=\frac{1}{4}\begin{bmatrix}3 & -1\\2 & -2\end{bmatrix}\begin{bmatrix}2 & -1\\2 & -3\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}3(2)+(-1)(2) & 3(-1)+(-1)(-3)\\2(2)+(-2)(2) & 2(-1)+(-2)(-3)\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}6-2 & -3+3\\4-4 & -2+6\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=I$$

Similarly:

$$B^{-1}B=\frac{1}{4}\begin{bmatrix}2 & -1\\2 & -3\end{bmatrix}\begin{bmatrix}3 & -1\\2 & -2\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}2(3)+(-1)(2) & 2(-1)+(-1)(-2)\\2(3)+(-3)(2) & 2(-1)+(-3)(-2)\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}4 & 0\\0 & 4\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=I$$

Hence: $BB^{-1} = I = B^{-1}B$

5. Determine whether the given matrices are multiplicative inverses of each other.

Ans. (i) $\begin{bmatrix}3 & 5\\4 & 7\end{bmatrix}$ and $\begin{bmatrix}7 & -5\\-4 & 3\end{bmatrix}$

$$\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}$$

$$=\begin{bmatrix} 3(7)+5(-4) & 3(-5)+5(3) \\ 4(7)+7(-4) & 4(-5(+7(3) \end{bmatrix}$$

$$=\begin{bmatrix} 21-20 & -15+15 \\ 28-28 & -20+21 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$$

$\therefore$ Given matrices are multiplicative inverse of each other.

(ii) $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 1(-3)+2(2) & 1(2)+2(-1) \\ 2(-3)+3(2) & 2(2)+3(-1) \end{bmatrix}$$

$$=\begin{bmatrix} -3+4 & 2-2 \\ -6+6 & 4-3 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I$$

6. **If** $A=\begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$, $B=\begin{bmatrix} -4 & -2 \\ 1 & -1 \end{bmatrix}$, $D=\begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$, **then verify that**

(i) $(AB)^{-1}=B^{-1}A^{-1}$

(ii) $(DA)^{-1}=A^{-1}D^{-1}$

Ans. **(i)** $\mathbf{(AB)^{-1}=B^{-1}A^{-1}}$

L.H.S $=(AB)^{-1}$

$$AB=\begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -4 & -2 \\ 1 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 4(-4)+0(1) & 4(-2)+0(-1) \\ -1(-4)+2(1) & -1(-2)+2(-1) \end{bmatrix}$$

$$=\begin{bmatrix} -16 & -8 \\ 6 & 0 \end{bmatrix}$$

$$|AB|=\begin{vmatrix} -16 & -8 \\ 6 & 0 \end{vmatrix}$$

$$=-16(0)-6(-8)$$

$$=0+48=48\neq 0$$

$$\text{Adj}(AB)=\begin{vmatrix} 0 & 8 \\ -6 & -16 \end{vmatrix}$$

$$(AB)^{-1}=\frac{1}{|AB|}\text{Adj}(AB)$$

$$=\frac{1}{48}\begin{bmatrix} 0 & 8 \\ -6 & -16 \end{bmatrix}=\begin{bmatrix} 0 & \frac{8}{48} \\ \frac{-6}{48} & \frac{-16}{48} \end{bmatrix}=\begin{bmatrix} 0 & 1/6 \\ \frac{-1}{8} & -1/3 \end{bmatrix}$$

R.H.S $=B^{-1}A^{-1}$

$$B=\begin{bmatrix} -4 & -2 \\ 1 & -1 \end{bmatrix}$$

$$|B|=-4(-1)-1)(-2)=4+2=6$$

$$B^{-1}=\frac{1}{|B|}\text{Adj}B=\frac{1}{6}\begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix}$$

$$A=\begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$$

$$|A|=\begin{vmatrix} 4 & 0 \\ -1 & 2 \end{vmatrix}=4(2)-(-1)(0)=8$$

$$A^{-1}=\frac{1}{|A|}\text{Adj}A=\frac{1}{8}\begin{bmatrix} 2 & 0 \\ .1 & 4 \end{bmatrix}$$

$$B^{-1}A^{-1}=\frac{1}{6}\begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix}\frac{1}{8}\begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$$

$$=\frac{1}{48}\begin{bmatrix} -1 & 2 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$$

$$=\frac{1}{48}\begin{bmatrix} -1(2)+2(1) & -1(0)+2(4) \\ -1(2)+-4(1) & -1(0)+-4(4) \end{bmatrix}$$

$$=\frac{1}{48}\begin{bmatrix} 0 & 8 \\ -6 & -16 \end{bmatrix}=\begin{bmatrix} 0 & \frac{8}{48} \\ -6/48 & -16/48 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 1/6 \\ -1/8 & -1/3 \end{bmatrix}$$

L.H.S = R.H.S

Hence: $(AB)^{-1}=B^{-1}A^{-1}$

(ii) $(DA)^{-1} = A^{-1}D^{-1}$

L.H.S $= (DA)^{-1}$

$$DA = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3(4)+1(-1) & -2(0)+1(2) \\ -2(4)+2(-1) & -2(0)+2(2) \end{bmatrix}$$

$$= \begin{bmatrix} 12-1 & 0+2 \\ -8-2 & 0+4 \end{bmatrix} = \begin{bmatrix} 11 & 2 \\ -10 & 4 \end{bmatrix}$$

$$|DA| = \begin{vmatrix} 11 & 2 \\ -10 & 4 \end{vmatrix}$$

$$= 11(4) - (-10)(2)$$

$$= 44 + 20$$

$$= 64$$

$$\text{Adj}(DA) = \begin{bmatrix} 4 & -2 \\ 10 & 11 \end{bmatrix}$$

$$(DA)^{-1} = \frac{1}{DA}\text{Adj}(DA)$$

$$= \frac{1}{64}\begin{bmatrix} 4 & -2 \\ 10 & 11 \end{bmatrix} = \begin{bmatrix} \frac{4}{64} & \frac{-2}{64} \\ \frac{10}{64} & \frac{11}{64} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{16} & \frac{-1}{32} \\ \frac{5}{32} & \frac{11}{64} \end{bmatrix}$$

R.H.S $= A^{-1}D^{-1}$

$$A = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 4 & 0 \\ -1 & 2 \end{vmatrix}$$

$$= 4(2) - (-1)(0)$$

$$= 8 \neq 0$$

$$A^{-1} = \frac{1}{|A|}\text{AdjA}$$

$$= \frac{1}{8}\begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$$

$$|D| = 3(2) - (-2)(1)$$

$$= 6 + 2 = 8$$

$$D^{-1} = \frac{1}{|D|}\text{AdjD}$$

$$= \frac{1}{8}\begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}$$

$$A^{-1}D^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \cdot \frac{1}{8}\begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}$$

$$A^{-1}D^{-1} = \frac{1}{64}\begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 3 \end{bmatrix}$$

$$= \frac{1}{64}\begin{bmatrix} 2(2)+0(2) & 2(-1)+0(3) \\ 1(2)+4(2) & 1(-1)+4(3) \end{bmatrix}$$

$$= \frac{1}{64}\begin{bmatrix} 4+0 & -2+0 \\ 2+8 & -1+12 \end{bmatrix}$$

$$= \frac{1}{64}\begin{bmatrix} 4 & -2 \\ 10 & 11 \end{bmatrix} = \begin{bmatrix} \frac{4}{64} & \frac{-2}{64} \\ \frac{10}{64} & \frac{11}{64} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{16} & \frac{-1}{32} \\ \frac{5}{32} & \frac{11}{64} \end{bmatrix}$$

L.H.S = R.H.S

Hence: $(DA)^{-1} = A^{-1}D^{-1}$

Solution of Simultaneous Linear Equations

System of two linear equations in two variables in general form is given as

$$ax + by = m$$

$$cx+dy=n$$

Where a, b, c, d, m and n are real numbers.

This system is also called simultaneous linear equations.

We discuss here the following methods of solution.

(i) **Matrix** inversion method.

(ii) **Cramer's** rule

(i) Matrix Inversion Method

Consider the system of linear questions

$$ax+by=m$$
$$cx+dy=n$$

$$\text{Then}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} m \\ n \end{bmatrix}$$

or $AX = B$

Where $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}, X=\begin{bmatrix} x \\ y \end{bmatrix}$ and $B=\begin{bmatrix} m \\ n \end{bmatrix}$

or $X = A^{-1}B$

$|A| = ad - bc$

$$\text{or } X=\frac{\text{Adj A}}{|A|}\times B$$

$$\because \quad A^{-1}=\frac{\text{Adj A}}{|A|} \text{ and } |A| \neq 0$$

$$\text{or}\begin{bmatrix} x \\ y \end{bmatrix}=\frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix}}{ad-bc}$$

$$=\begin{bmatrix} \dfrac{dm-bn}{ad-bc} \\ \dfrac{-cm+an}{ad-bc} \end{bmatrix}$$

$$\Rightarrow \quad x=\frac{dm-bn}{ad-bc} \text{ and } y=\frac{an-cm}{ad-bc}$$

(ii) Cramer's Rule.

Consider the following system of linear equations.

$$ax+by=m$$
$$cx+dy=n$$

We know that

$AX = B$, where $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X=\begin{bmatrix} x \\ y \end{bmatrix}$

and $B=\begin{bmatrix} m \\ n \end{bmatrix}$

or $\quad X = A^{-1}B$

$$\text{or } \quad X=\frac{\text{Adj A}}{|A|}\times B$$

$$\text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix}=\frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix}}{|A|}$$

$$=\frac{\begin{bmatrix} dm-bn \\ -cm+an \end{bmatrix}}{|A|}$$

$$=\begin{bmatrix} \dfrac{dm-bn}{|A|} \\ \dfrac{-cm+an}{|A|} \end{bmatrix}$$

$$\text{or} \quad x=\frac{dm-bn}{|A|}=\frac{|A_x|}{|A|}$$

$$\text{and} \quad y=\frac{an-cm}{|A|}=\frac{|A_y|}{|A|}$$

where $|A_x|=\begin{vmatrix} m & b \\ n & d \end{vmatrix}$ and

$$|A_y|=\begin{vmatrix} a & m \\ c & n \end{vmatrix}$$

Example 1

Solve the following system by using matrix inversion method.

$$4x - 2y = 8$$
$$3x + y = -4$$

Solution

Step I $\begin{bmatrix}4 & -2\\3 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}8\\-4\end{bmatrix}$

Step 2

The coefficient matrix $M=\begin{bmatrix}4 & -2\\3 & 1\end{bmatrix}$ is non-singular, since

$\det M = 4\times 1-3(-2)=4+6=10\neq 0$. So M^{-1} is possible.

Step 3

$$\begin{bmatrix}x\\y\end{bmatrix}=M^{-1}\begin{bmatrix}8\\-4\end{bmatrix}$$

$$=\frac{1}{10}\begin{bmatrix}1 & 2\\-3 & 4\end{bmatrix}\begin{bmatrix}8\\-4\end{bmatrix}$$

$$=\frac{1}{10}\begin{bmatrix}8-8\\-24-16\end{bmatrix}$$

$$=\frac{1}{10}\begin{bmatrix}0\\-40\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\-4\end{bmatrix}$$

$\Rightarrow$ $x=0$ and $y=-4$

Example 2

Solve the following system of linear equations by using Cramer's rule.

$$3x-2y=1$$
$$-2x+3y=2$$

Solution

$$3x-2y=1$$
$$-2x+3y=2$$

We have

$$A=\begin{bmatrix}3 & -2\\-2 & 3\end{bmatrix},$$

$$A_x=\begin{bmatrix}1 & -2\\2 & 3\end{bmatrix},$$

$$A_y=\begin{bmatrix}3 & 1\\-2 & 2\end{bmatrix}$$

$|A|=\begin{vmatrix}3 & -2\\-2 & 3\end{vmatrix}=9-4=5\neq 0$ (non-singular)

$$x=\frac{|A_x|}{|A|}=\frac{\begin{vmatrix}1 & -2\\2 & 3\end{vmatrix}}{5}=\frac{3+4}{5}=\frac{7}{5}$$

$$y=\frac{|A_y|}{|A|}=\frac{\begin{vmatrix}3 & 1\\-2 & 2\end{vmatrix}}{5}=\frac{6+2}{5}=\frac{8}{5}$$

$$S.S=\left\{\left(\frac{7}{5},\frac{8}{5}\right)\right\}$$

Example 3

The length of a rectangle is 6 cm less than three times its width. The perimeter of the rectangle is 140 cm. Find the dimensions of the rectangle.

(by using matrix inversion method)

Solution

If width of the rectangle is x cm, then length of the rectangle ycm

According to first condition

$$y=3x-6,$$

According to 2^{nd} condition

The perimeter $=2x+2y=140$

$\Rightarrow$ $x+y=70$ .... (i)

and $3x-y=6$ .... (ii)

In the matrix form

$$\begin{bmatrix}1 & 1\\3 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}70\\6\end{bmatrix}$$

$$\det\begin{bmatrix}1 & 1\\3 & -1\end{bmatrix}=\begin{vmatrix}1 & 1\\3 & -1\end{vmatrix}$$

$$=1\times(-1)-3\times 1=-1-3=-4\neq 0$$

We know that:

$$X=A^{-1}B \text{ and } A^{-1}=\frac{\text{Adj } A}{|A|}$$

Hence $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-4}\begin{bmatrix} -1 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 70 \\ 6 \end{bmatrix}$

$= \frac{-1}{4}\begin{bmatrix} -70-6 \\ -210+6 \end{bmatrix} \qquad = \begin{bmatrix} \frac{76}{4} \\ \frac{204}{4} \end{bmatrix} = \begin{bmatrix} 19 \\ 51 \end{bmatrix}$

Thus, by the equality of matrices, width of the rectangle x = 19 cm and the length y = 51 cm.

Exercise 1.6

1. Use matrices, if possible, to solve the following systems of linear equations by:

(i) the matrix inverse method

(ii) the Cramer's rule.

(i) $2x-2y=4$

$3x+2y=6$

Matrix inverse method

$$\begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$A = \begin{bmatrix} 2 & -2 \\ 3 & 2 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \end{bmatrix}$; $B = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$

A X = B

$X = A^{-1}B$ ....(i)

$$|A| = \begin{vmatrix} 2 & -2 \\ 3 & 2 \end{vmatrix}$$

$= 2(2)-(-2)(3)$

$= 4 + 6 = 10 \neq 0$

As $|A| \neq 0$ so solution is possible

$$\text{Adj } A = \begin{bmatrix} 2 & 2 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} AdjA$$

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 2 & 2 \\ -3 & 2 \end{bmatrix}$$

Putting the values of A^{-1} and B in equation (i)

$$X = \frac{1}{10}\begin{bmatrix} 2 & 2 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$$X = \frac{1}{10}\begin{bmatrix} 2(4)+2(6) \\ -3(4)+2(6) \end{bmatrix}$$

$$X = \frac{1}{10}\begin{bmatrix} 8+12 \\ -12+12 \end{bmatrix}$$

$$X = \frac{1}{10}\begin{bmatrix} 20 \\ 0 \end{bmatrix}$$

$$X = \begin{bmatrix} \cancel{20}^{2} \times \frac{1}{\cancel{10}} \\ 0 \times \frac{1}{10} \end{bmatrix}$$

$$X = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $x=2$

$y=0$

$S.S. = \{(x,y)\} = \{(2,0)\}$

$S.S. = \{(2,0)\}$

(ii) $2x+y=3$

$6x+5y=1$

In matrices form

$$\begin{bmatrix}2 & 1\\6 & 5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}$$

Let

$$A=\begin{bmatrix}2 & 1\\6 & 5\end{bmatrix};\ X=\begin{bmatrix}x\\y\end{bmatrix};B=\begin{bmatrix}3\\1\end{bmatrix}$$

$AX=B$

$X=A^{-1}B$

$$|A|=\begin{vmatrix}2 & 1\\6 & 5\end{vmatrix}$$

$$=2(5)-6(1)$$

$$=10-6$$

$$|A|=4\neq 0$$

As $|A|\neq 0$, so solution is possible

$$Adj\,A=\begin{bmatrix}5 & -1\\-6 & 2\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}\times Adj\,A$$

$$A^{-1}=\frac{1}{4}\begin{bmatrix}5 & -1\\-6 & 2\end{bmatrix}$$

Putting the value of A^{-1} & B in equation i.

$$X=A^{-1}B$$

$$=\frac{1}{4}\begin{bmatrix}5 & -1\\-6 & 2\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}5(3)+(-1)(1)\\-6(3)+2(1)\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}15-1\\-18+2\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}14\\-16\end{bmatrix}$$

$$=\begin{bmatrix}\frac{14}{4}\\\frac{16}{4}\end{bmatrix}$$

$$x=\begin{bmatrix}\frac{7}{2}\\-4\end{bmatrix}$$

$$\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{7}{2}\\-4\end{bmatrix}$$

$$\Rightarrow\quad x=\frac{7}{2}$$

$$y=-4$$

Solution set $S.S.=\left\{\left(\frac{7}{2},-4\right)\right\}$

(iii) $4x+2y=8$

$3x-y=-1$

In matrices form

$$\begin{bmatrix}4 & 2\\3 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}8\\-1\end{bmatrix}$$

Let

$$A=\begin{bmatrix}4 & 2\\3 & -1\end{bmatrix};X=\begin{bmatrix}x\\y\end{bmatrix};B=\begin{bmatrix}8\\-1\end{bmatrix}$$

$AX=B$

$X=A^{-1}B$

$$|A|=\begin{vmatrix}4 & 2\\3 & -1\end{vmatrix}$$

$$=4(-1)-3(2)$$

$$=-4-6$$

$$|A|=-10\neq 0$$

As $|A|\neq 0$, so solution is possible

$$\text{Adj } A=\begin{bmatrix}-1 & -2\\ -3 & 4\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}\times \text{Adj} A$$

$$A^{-1}=\frac{1}{-10}\begin{bmatrix}-1 & -2\\ -3 & 4\end{bmatrix}$$

Putting values of A^{-1} & B in equation.

$$X=A^{-1}B$$

$$X=\frac{1}{-10}\begin{bmatrix}-1 & -2\\ -3 & 4\end{bmatrix}\begin{bmatrix}8\\ -1\end{bmatrix}$$

$$X=\frac{1}{-10}\begin{bmatrix}-1(8)+(-2)(-1)\\ -3(8)+4(-1)\end{bmatrix}$$

$$X=\frac{1}{-10}\begin{bmatrix}-8+2\\ -24-4\end{bmatrix}$$

$$=\frac{1}{-10}\begin{bmatrix}-6\\ -28\end{bmatrix}$$

$$=\begin{bmatrix}-\cancel{6}^{3}\times\frac{1}{\cancel{-10}_{5}}\\ -\cancel{28}^{14}\times\frac{1}{\cancel{-10}_{5}}\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}\frac{3}{5}\\ \frac{14}{5}\end{bmatrix}$$

$$\Rightarrow \quad x=\frac{3}{5}$$

$$y=\frac{14}{5}$$

$$S.S=\left\{\left(\frac{3}{5},\frac{14}{5}\right)\right\}$$

(iv) $3x-2y=-6$

$5x-2y=-10$

In matrices form

$$\begin{bmatrix}3 & -2\\ 5 & -2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-6\\ -10\end{bmatrix}$$

Let

$$A=\begin{bmatrix}3 & -2\\ 5 & -2\end{bmatrix};\ X=\begin{bmatrix}x\\ y\end{bmatrix};B=\begin{bmatrix}-6\\ -10\end{bmatrix}$$

$$AX = B$$

$$\Rightarrow \quad X=A^{-1}B$$

$$|A|=\begin{vmatrix}3 & -2\\ 5 & -2\end{vmatrix}$$

$$=3(-2)-(5)(2)$$

$$=-6+10$$

$$|A|=4\neq 0$$

As $|A|\neq 0$, so solution is possible

$$\text{Adj } A=\begin{bmatrix}-2 & 2\\ -5 & 3\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}\times \text{Adj} A$$

$$A^{-1}=\frac{1}{4}\begin{bmatrix}-2 & 2\\ -5 & 3\end{bmatrix}$$

Putting the values of A^{-1} & B in equation i.

$$X=A^{-1}B$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{4}\begin{bmatrix}-2 & 2\\ -5 & 3\end{bmatrix}\begin{bmatrix}-6\\ -10\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{4}\begin{bmatrix}-2(-6)+2(-10)\\ -5(-6)+3(-10)\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}12+(-20)\\ 30-30\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}-8\\ 0\end{bmatrix}$$

$$=\begin{bmatrix} \cancel{-8}^{2}\times\frac{1}{\cancel{4}} \\ 0\times\frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x=-2$

$y=0$

$S.S=\{(-2,0)\}$

(v) $3x-2y=4$

$-6x+4y=7$

In matrices form

$$\begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Let

$$A=\begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix};X=\begin{bmatrix} x \\ y \end{bmatrix};B=\begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$AX=B$

$X=A^{-1}B$

$$|A|=\begin{bmatrix} 3 & -2 \\ -6 & 4 \end{bmatrix}$$

$=3(4)-(6)(-2)$

$=12-12$

$=0$

As $|A|=0$, so solution is not possible

(vi) $4x+y=9$

$-3x-y=-5$

In matrices form

$$\begin{bmatrix} 4 & 1 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 9 \\ -5 \end{bmatrix}$$

Let

$$A=\begin{bmatrix} 4 & 1 \\ -3 & -1 \end{bmatrix};X=\begin{bmatrix} x \\ y \end{bmatrix};B=\begin{bmatrix} 9 \\ -5 \end{bmatrix}$$

$AX=B$

$X=A^{-1}B$

$$|A|=\begin{vmatrix} 4 & 1 \\ -3 & -1 \end{vmatrix}$$

$=4(-1)-(-3)(1)$

$=-4+3$

$=-1\neq 0$

As $|A|\neq 0$, so solution is possible

$$\text{Adj } A=\begin{bmatrix} -1 & -1 \\ 3 & 4 \end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}\times Adj\,A$$

$$=\frac{1}{-1}\times\begin{bmatrix} -1 & -1 \\ 3 & 4 \end{bmatrix}$$

Putting the values in equation (i) of A^{-1} and B

$$X=A^{-1}B$$

$$X=\frac{1}{-1}\begin{bmatrix} -1 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 9 \\ -5 \end{bmatrix}$$

$$=\frac{1}{-1}\begin{bmatrix} -1(9)+(-1)(-5) \\ 3(9)+4(-5) \end{bmatrix}$$

$$=\frac{1}{-1}\begin{bmatrix} -9+5 \\ 27-20 \end{bmatrix}$$

$$=\frac{1}{-1}\begin{bmatrix} -4 \\ 7 \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{-1}\times-4 \\ \frac{1}{-1}\times 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$$

$$\Rightarrow \quad x = 4$$

$$y = -7$$

$$S.S. = \{(4, -7)\}$$

(vii) $2x - 2y = 4$

$-5x - 2y = -10$

In matrices form

$$\begin{bmatrix} 2 & -2 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \end{bmatrix}$$

$A X = B$

$\Rightarrow X = A^{-1} B$

Let $A = \begin{bmatrix} 2 & -2 \\ -5 & -2 \end{bmatrix}; X = \begin{bmatrix} x \\ y \end{bmatrix}; B = \begin{bmatrix} 4 \\ -10 \end{bmatrix}$

$AX = B$

$X = A^{-1}B$

$$|A| = \begin{vmatrix} 2 & -2 \\ -5 & -2 \end{vmatrix}$$

$$= 2(-2) - (-5)(-2)$$

$$= -4 - 10$$

$$|A| = -14 \neq 0$$

As $|A| \neq 0$, so solution is possible

$$\text{Adj } A = = \begin{bmatrix} -2 & 2 \\ 5 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \times \text{Adj } A$$

$$= \frac{1}{-14} \times \begin{bmatrix} -2 & 2 \\ 5 & 2 \end{bmatrix}$$

Putting the values of A^{-1} and B in equation

(i) $X = A^{-1} B$

$$X = \frac{1}{-14} \times \begin{bmatrix} -2 & 2 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -10 \end{bmatrix}$$

$$X = \frac{1}{-14} \begin{bmatrix} -2(4) + 2(-10) \\ 5(4) + 2(-10) \end{bmatrix}$$

$$= \frac{1}{-14} \begin{bmatrix} -8 - 20 \\ 20 - 20 \end{bmatrix}$$

$$= \frac{1}{-14} \begin{bmatrix} -28 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\cancel{28}^{2} \times \frac{1}{\cancel{-14}} \\ 0 \times \frac{1}{-14} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x = 2$$

$$y = 0$$

$$S.S. = \{(2, 0)\}$$

(viii) $3x - 4y = 4$

$x + 2y = 8$

In matrices form

$$\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Let

$A = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}; X = \begin{bmatrix} x \\ y \end{bmatrix}; B = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$AX = B$

$\Rightarrow \quad X = A^{-1} B$ i

$$|A| = \begin{vmatrix} 3 & -4 \\ 1 & 2 \end{vmatrix}$$

$$= 3(2) - (1)(-4)$$

$$= 6 + 4$$

$$|A| = 10 \neq 0$$

As $|A| \neq 0$, so solution is possible

$$\text{Adj} A=\begin{bmatrix}2 & 4\\ -1 & 3\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}\times \text{Adj} A$$

$$A^{-1}=\frac{1}{10}\times\begin{bmatrix}2 & 4\\ -1 & 3\end{bmatrix}$$

Putting the values of A^{-1} & B in equation (i)

$$X=A^{-1}B$$

$$X=\frac{1}{10}\begin{bmatrix}2 & 4\\ -1 & 3\end{bmatrix}\begin{bmatrix}4\\ 8\end{bmatrix}$$

$$X=\frac{1}{10}\begin{bmatrix}2(4)+4(8)\\ -1(4)+3(8)\end{bmatrix}$$

$$X=\frac{1}{10}\begin{bmatrix}8+32\\ -4+24\end{bmatrix}$$

$$X=\frac{1}{10}\begin{bmatrix}40\\ 20\end{bmatrix}$$

$$X=\begin{bmatrix}\cancel{40}^{4}\times\frac{1}{\cancel{10}}\\ \cancel{20}^{2}\times\frac{1}{\cancel{10}}\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}4\\ 2\end{bmatrix}$$

$\Rightarrow$ $x=4$

$y=2$

$S.S.=\{(4,2)\}$

Cramer's rule

(i) $2x-2y=4$

$3x+2y=6$

In matrices form

$$\begin{bmatrix}2 & -2\\ 3 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}4\\ 6\end{bmatrix}$$

Let $A=\begin{bmatrix}2 & -2\\ 3 & 2\end{bmatrix}$; $X=\begin{bmatrix}x\\ y\end{bmatrix}$; $B=\begin{bmatrix}4\\ 6\end{bmatrix}$

$$|A|=\begin{vmatrix}2 & -2\\ 3 & 2\end{vmatrix}$$

$$=2(2)-3(-2)$$

$$=4+6$$

$$|A|=10\neq 0$$

As $|A|\neq 0$, so solution is possible.

A_x; - (Determinant No. 1)

In determinant 1 we change first column to constant matrix.

$$|A_x|=\begin{vmatrix}4 & -2\\ 6 & 2\end{vmatrix}$$

$$=4(2)-6(-2)$$

$$=8+12$$

$$|A_x|=20$$

$$x=\frac{|A_x|}{|A|}=\frac{20}{10}=2$$

$$x=2$$

$|A_y|$ (Determinant No.2)

In determinant 2 we change 2^{nd} column to constant matrix.

$$|A_y|=\begin{vmatrix}2 & 4\\ 3 & 6\end{vmatrix}$$

$$=2(6)-3(4)$$

$$=12-12$$

$$|A_y|=0$$

$$y=\frac{|A_y|}{|A|}=\frac{0}{10}=0$$

$$y=0$$

S.S=$\{(2, 0)\}$ .ans.

(ii) $2x+y=3$

$6x+5y=1$

In matrices form

$$\begin{bmatrix}2 & 1\\6 & 5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}3\\1\end{bmatrix}$$

Let

$$A=\begin{bmatrix}2 & 1\\6 & 5\end{bmatrix};X=\begin{bmatrix}x\\y\end{bmatrix};B=\begin{bmatrix}3\\1\end{bmatrix}$$

$$|A|=\begin{vmatrix}2 & 1\\6 & 5\end{vmatrix}$$

$$=2(5)-6(1)$$

$$=10-6$$

$$|A|=4\neq 0$$

As $|A|\neq 0$, so solution is possible.

$$|A_x|=\begin{vmatrix}3 & 1\\1 & 5\end{vmatrix}$$

$$=3(5)-1(1)$$

$$|A_x|=15-1$$

$$|A_x|=14$$

$$x=\frac{|A_x|}{|A|}=\frac{\cancel{14}^{7}}{\cancel{4}^{2}}$$

$$x=\frac{7}{2}$$

$$|A_y|=\begin{vmatrix}2 & 3\\6 & 1\end{vmatrix}$$

$$=2(1)-6(3)$$

$$|A_y|=2-18$$

$$|A_y|=-16$$

$$y=\frac{|A_y|}{|A|}=\frac{-16}{4}=-4$$

$$y=-4$$

$$S.S=\left\{\left(\frac{7}{2},-4\right)\right\}$$

(iii) $4x+2y=8$

$3x-y=-1$

In matrices form

$$\begin{bmatrix}4 & 2\\3 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}8\\-1\end{bmatrix}$$

Let

$$A=\begin{bmatrix}4 & 2\\3 & -1\end{bmatrix};X=\begin{bmatrix}x\\y\end{bmatrix};B=\begin{bmatrix}8\\-1\end{bmatrix}$$

$$|A|=4(-1)-3(2)$$

$$=-4-6$$

$$|A|=-10\neq 0$$

As $|A|\neq 0$, so solution is possible.

$$|A_x|=\begin{vmatrix}8 & 2\\-1 & -1\end{vmatrix}$$

$$=8(-1)-2(-1)$$

$$=-8+2$$

$$=-6$$

$$x=\frac{|A_x|}{|A|}$$

$$x=\frac{-\cancel{6}^{3}}{-\cancel{10}5}=\frac{3}{5}$$

$$|A_y|=\begin{vmatrix}4 & 8\\3 & -1\end{vmatrix}$$

$$=4(-1)-(3)(8)$$

$$=-4-24$$

$$=-28$$

$$y=\frac{|A_y|}{|A|}$$

$$=\frac{-28}{-10}=\frac{14}{5}$$

$$S.S.=\left\{\left(\frac{3}{5},\frac{14}{5}\right)\right\}$$

(iv) $3x-2y=-6$

$5x-2y=-10$

In matrices form

$$\begin{bmatrix}3 & -2\\5 & -2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}-6\\-10\end{bmatrix}$$

Let

$$A=\begin{bmatrix}3 & -2\\5 & -2\end{bmatrix};X=\begin{bmatrix}x\\y\end{bmatrix};B=\begin{bmatrix}-6\\-10\end{bmatrix}$$

$$|A|=\begin{vmatrix}3 & -2\\5 & -2\end{vmatrix}$$

$=3(-2)-5(-2)$

$=-6+10$

$|A|=4\neq 0$

As $|A|\neq 0$, so solution is possible.

$$|A_x|=\begin{vmatrix}-6 & -2\\-10 & -2\end{vmatrix}$$

$=-6(-2)-(-2)(-10)$

$=12-20$

$|A_x|=-8$

$$x=\frac{|A_x|}{|A|}=\frac{-\cancel{8}^{2}}{\cancel{4}}$$

$x=-2$

$$|A_y|=\begin{vmatrix}3 & -6\\5 & -10\end{vmatrix}$$

$=3(-10)-(5)(-6)$

$=-30+30$

$=0$

$$y=\frac{|A_y|}{|A|}=\frac{0}{4}$$

$y=0$

$S.S.=\{(-2,0)\}$

(v) $3x-2y=4$

$-6x+4y=7$

In matrices form

$$\begin{bmatrix}3 & -2\\-6 & 4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}4\\7\end{bmatrix}$$

Let

$$A=\begin{bmatrix}3 & -2\\-6 & 4\end{bmatrix};X=\begin{bmatrix}x\\y\end{bmatrix};B=\begin{bmatrix}4\\7\end{bmatrix}$$

$$|A|=\begin{vmatrix}3 & -2\\-6 & 4\end{vmatrix}$$

$=3(4)-(-6)(-2)$

$=12-12$

$|A|=0$

As $|A|=0$, so solution is not possible

(vi) $4x+y=9$

$-3x-y=-5$

In matrices form

$$\begin{bmatrix}4 & 1\\-3 & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}9\\-5\end{bmatrix}$$

Let

$$A=\begin{bmatrix}4 & 1\\-3 & -1\end{bmatrix};X=\begin{bmatrix}x\\y\end{bmatrix};B=\begin{bmatrix}9\\-5\end{bmatrix}$$

$$|A|=\begin{vmatrix}4 & 1\\-3 & -1\end{vmatrix}$$

$=4(-1)-(-3)(1)$

$=-4+3$

$|A|=-1\neq 0$

As $|A|\neq 0$, so solution is possible.

$$|A_x|=\begin{vmatrix}9 & 1\\-5 & -1\end{vmatrix}$$

$= 9(-1) - 1(-5)$

$= -4$

$x = \frac{|A_x|}{|A|} = \frac{\cancel{-}4}{\cancel{-}1}$

$x = 4$

$|A_y| = \begin{vmatrix} 4 & 9 \\ -3 & -5 \end{vmatrix}$

$= 4(-5) - 9(-3)$

$= -20 + 27$

$= 7$

$y = \frac{|A_y|}{|A|} = \frac{7}{-1}$

$y = -7$

$S.S = \{(4, -7)\}$

(vii) $2x - 2y = 4$

$-5x - 2y = -10$

In matrices form

$$\begin{bmatrix} 2 & -2 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \end{bmatrix}$$

Let

$A = \begin{bmatrix} 2 & -2 \\ -5 & -2 \end{bmatrix}; X = \begin{bmatrix} x \\ y \end{bmatrix}; B = \begin{bmatrix} 4 \\ -10 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & -2 \\ -5 & -2 \end{vmatrix}$

$= 2(-2) - (-5)(-2)$

$= -4 - 10$

$|A| = -14 \neq 0$

As $|A| \neq 0$, so solution is possible.

$|A_x| = \begin{vmatrix} 4 & -2 \\ -10 & -2 \end{vmatrix}$

$= 4(-2) - (-10)(-2)$

$= -8 - 20$

$= -28$

$x = \frac{|A_x|}{|A|} = \frac{\cancel{-}\cancel{28}^{2}}{\cancel{-}\cancel{14}}$

$x = 2$

$|A_y| = \begin{vmatrix} 2 & 4 \\ -5 & -10 \end{vmatrix}$

$= 2(-10) - (-5)(4)$

$= -20 + 20$

$= 0$

$y = \frac{|A_y|}{|A|} = \frac{0}{-14}$

$y = 0$

$S.S = \{(2, 0)\}$ ans.

(viii) $3x - 4y = 4$

$x + 2y = 8$

In matrices form

$$\begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Let

$A = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}; X = \begin{bmatrix} x \\ y \end{bmatrix}; B = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$

$|A| = \begin{vmatrix} 3 & -4 \\ 1 & 2 \end{vmatrix}$

$= 3(2) - 1(-4)$

$= 6 + 4$

$|A| = 10 \neq 0$

As $|A| \neq 0$, so solution is possible.

$|A_x| = \begin{vmatrix} 4 & -4 \\ 8 & 2 \end{vmatrix}$

$= 4(2) - 8(-4)$

$= 8 + 32$

$= 40$

$$x=\frac{|A_x|}{|A|}=\frac{\cancel{40}\,4}{\cancel{10}}$$

$$x=4$$

$$|A_y|=\begin{vmatrix}3 & 4\\ 1 & 8\end{vmatrix}$$

$$=3(8)-1(4)$$

$$=24-4$$

$$=20$$

$$y=\frac{|A_y|}{|A|}=\frac{\cancel{20}^{\,2}}{\cancel{10}}$$

$$y=2$$

$S.S.=\{(4,2)\}$ ans.

Q.2. The length of a rectangle is 4 times its width. The perimeter of the rectangle is 150cm. Find dimensions of the rectangle?

Let width of rectangle $= x$.

and length of rectangle $= y$

According to first condition

$$y=4x$$

$$4x-y=0.........(i)$$

According to 2nd condition

Perimeter $=150$cm.

$$2(x+y)=150$$

$$x+y=\frac{150}{2}$$

$$x+y=75\ ....(ii)$$

In matrices form

$$\begin{bmatrix}1 & 1\\ 4 & -1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}75\\ 0\end{bmatrix}$$

$$AX=B$$

$$\Rightarrow \quad X=A^{-1}B$$

Now

$$A=\begin{bmatrix}1 & 1\\ 4 & -1\end{bmatrix}$$

$$|A|=\begin{vmatrix}1 & 1\\ 4 & -1\end{vmatrix}$$

$$=1(-1)-4(1)$$

$$=-1-4$$

$$=-5\neq 0$$

$$Adj\ A=\begin{bmatrix}-1 & -1\\ -4 & 1\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}\times \text{Adj}\,A$$

$$A^{-1}=-\frac{1}{5}\begin{bmatrix}-1 & -1\\ -4 & 1\end{bmatrix}$$

$$=\frac{1}{5}\begin{bmatrix}1 & 1\\ 4 & -1\end{bmatrix}$$

$$\Rightarrow \quad X=A^{-1}B$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{5}\begin{bmatrix}1 & 1\\ 4 & -1\end{bmatrix}\begin{bmatrix}75\\ 0\end{bmatrix}$$

$$=\frac{1}{5}\begin{bmatrix}1(75)+1(0)\\ 4(75)+(-1)(0)\end{bmatrix}$$

$$=\frac{1}{5}\begin{bmatrix}75\\ 300\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}\frac{75}{5}\\ \frac{300}{5}\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}15\\ 60\end{bmatrix}$$

$$\Rightarrow \quad x=15cm$$

$$\Rightarrow \quad y=60cm$$

Q.3. Two sides of rectangle differ by 3.5cm. Find the dimensions of the rectangle if its perimeter is 67cm.

Let required sides of rectangle are x and y.

According to first condition

$$x-y=3.5 \longrightarrow (i)$$

According to 2[nd] condition

Perimeter $=67$

$2(x+y) = 67$

$$\Rightarrow \quad x+y = 33.5 \longrightarrow (ii)$$

In matrices form

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 3.5 \\ 33.5 \end{bmatrix}$$

We have

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}; A_x = \begin{bmatrix} 3.5 & -1 \\ 33.5 & 1 \end{bmatrix},$$

$$A_y = \begin{bmatrix} 1 & 3.5 \\ 1 & 33.5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 1(1)-1(-1)$$

$$= 1+1 = 2 \neq 0$$

$$x = \frac{|A_x|}{|A|}$$

$$= \frac{\begin{vmatrix} 3.5 & -1 \\ 33.5 & 1 \end{vmatrix}}{2}$$

$$= \frac{3.5(1)-33.5(-1)}{2}$$

$$= \frac{3.5+33.5}{2}$$

$$= \frac{37}{2} = 18.5$$

$$y = \frac{|A_y|}{|A|}$$

$$= \frac{\begin{vmatrix} 1 & 3.5 \\ 1 & 33.5 \end{vmatrix}}{2}$$

$$= \frac{1(33.5)-1(3.5)}{2}$$

$$= \frac{33.5-3.5}{2}$$

$$= \frac{30}{2} = 15$$

$$\Rightarrow \quad x=18.5, \qquad y=15$$

Q.4. The third angle of an isosceles triangle is 16° less than the sum of the two equal angles. Find three angles of the triangle.

Let third angle of triangle = y

and two equal angle of triangle = x

we know that

$$x+x+y = 180^o$$

$$2x+y = 180^o \ldots\ldots\ldots\ldots(i)$$

According to given condition.

$$y = 2x-16$$

$$2x-y = 16$$

In matrices form

$$\begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 180 \\ 16 \end{bmatrix}$$

$$AX = B$$

$$\Rightarrow \quad X = A^{-1}B$$

Now

$$A = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix}$$

$$|A| = 2(-1)-2(1)$$

$$= -2-2$$
$$= -4 \neq 0$$

$$\text{Adj} A = \begin{bmatrix} -1 & -1 \\ -2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \times \text{Adj A}$$

$$= \frac{1}{-4} \begin{bmatrix} -1 & -1 \\ -2 & 2 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 180 \\ 16 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1(180)+1(16) \\ 2(180)+(-2)(16) \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 180+16 \\ 360-32 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 196 \\ 328 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 49 \\ 82 \end{bmatrix}$$

Hence: $x = 49^\circ$, $y = 82^\circ$

Required angles are 49°, 49°, 82°.

Q.5. One acute angle of a right triangle is 12° more than twice the other acute angle. Find the acute angles of the right triangle?

Let acute angles of right angled triangle are x and y

We know that

$$x + y = 90^o \ (i)$$

According to given condition

$$x = 2y + 12^o$$

$$x - 2y = 12^o \longrightarrow (ii)$$

In matrix form

$$\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 90 \\ 12 \end{bmatrix}$$

We have

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, A_x = \begin{bmatrix} 90 & 1 \\ 12 & -2 \end{bmatrix}$$

$$A_y = \begin{bmatrix} 1 & 90 \\ 1 & 12 \end{bmatrix}$$

Now $$A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$|A| = 1(-2) - 1(1)$$
$$= -2-1$$
$$= -3 \neq 0$$

$$x = \frac{|A_x|}{|A|}$$

$$= \frac{\begin{vmatrix} 90 & 1 \\ 12 & -2 \end{vmatrix}}{-3}$$

$$= \frac{90(-2) - 1(12)}{-3}$$

$$x = \frac{-180-12}{-3}$$

$$= \frac{-192}{-3} = 64^o$$

$$y = \frac{|A_y|}{|A|}$$

$$y = \frac{\begin{vmatrix} 1 & 90 \\ 1 & 12 \end{vmatrix}}{-3}$$

$$= \frac{1(12)-1(90)}{-3}$$

$$= \frac{12-90}{-3}$$

$$= \frac{-78}{-3}$$

$$= 26^o$$

$\therefore$ Required angles are 26° and 64°

$\Rightarrow \quad x=64^o$

$\Rightarrow \quad y=26^o$

Q6. Two cars that are 600 km apart are moving towards each other. Their speeds differ by 6km per hour and the cars are 123 km apart after $4\frac{1}{2}$ hours. Find the speed of each car.

Solution:

Let required speed of two cars are x and y

According to given condition

$$x-y=6$$

$$\frac{9}{2}x-\frac{9}{2}y=600-123=477$$

$$x-y=6$$

$$9x+9y=477\times 2=954$$

$$\Rightarrow \quad x-y=6$$
$$9x+9y=954$$

In matrix form

$$\begin{bmatrix} 1 & -1 \\ 9 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 6 \\ 954 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 \\ 9 & 9 \end{bmatrix}, A_x = \begin{bmatrix} 6 & -1 \\ 954 & 9 \end{bmatrix}$$

$$A_y = \begin{bmatrix} 1 & 6 \\ 9 & 954 \end{bmatrix}$$

Now

$$A = \begin{bmatrix} 1 & -1 \\ 9 & 9 \end{bmatrix}$$

$$|A| = 1(9)-(-1)(9)$$
$$= 9+9=0$$
$$= 18 \neq 0$$

$$x = \frac{|A_x|}{|A|} = \frac{\begin{vmatrix} 6 & -1 \\ 954 & 9 \end{vmatrix}}{18}$$

$$= \frac{6(9)-(-1)(954)}{18} = \frac{54+954}{18} = \frac{1008}{18} = 56\text{km/h}$$

$$y = \frac{|A_y|}{|A|} = \frac{\begin{vmatrix} 1 & 6 \\ 9 & 954 \end{vmatrix}}{18}$$

$$= \frac{1(954)-6(9)}{18}$$

$$= \frac{954-54}{18}$$

$$= \frac{900}{18} = 50km/h$$

OBJECTIVE

1 .The order of matrix [2 1] is ....

(a) 2-by-1 (b) 1-by-2
(c) 1-by-1 (d) 2-by-2

2. $\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}$ is called .... Matrix.

(a) zero (b) unit
(c) scalar (d) singular

3. Which is order of a square matrix ?

(a) 2-by-2 (b) 1-by-2
(c) 2-by-1 (d) 3-by-2

4. Which is order of a rectangular matrix?

(a) 2-by-2 (b) 4-by-4
(c) 2-by-1 (d) 3-by-3

5. Order of transpose of $\begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 3 & 2 \end{bmatrix}$ is ...:

(a) 3-by-2 (b) 2-by-3
(c) 1-by-3 (d) 3-by-1

6. Adjoint of $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ is ....

(a) $\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$

7. If $\begin{vmatrix} 2 & 6 \\ 3 & x \end{vmatrix} = 0$, then x is equal to:

(a) 9 (b) −6
(c) 6 (d) −9

8. Product of $[x \quad y]\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is ....

(a) $[2x+y]$ (b) $[x-2y]$
(c) $[2x-y]$ (d) $[x+2y]$

9. If $x+\begin{bmatrix} -1 & -2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

then x is equal to.........

(a) $\begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$

10. The idea of a matrices was given by:__

(a) Arthur Cayley (b) Dr. Aslam
(c) Dr. Ali (d) Dr. Khalid

11. The matrix $M = [2 \ -1 \ 7]$ is a---- matrix.

(a) Row (b) Column
(c) Square (d) Null

12. The matrix $N = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ is a ____ matrix.

(a) Row (b) Column
(c) Square (d) Null

13. The matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$ is a __ matrix.

(a) Rectangular (b) Square
(c) Row (d) Column

14. The matrix $B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$ is a __ matrix.

(a) Rectangular (b) Square
(c) Row (d) Column

15. If A is a matrix then its transpose is denoted by:

(a) A^c (b) A^t
(c) A (d) $(A^t)^t$

16. If $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ then $-A =$ _______

(a) $\begin{bmatrix} -1 & 2 \\ -3 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 \\ -3 & -4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$

17. A square matrix is symmetric if _____

(a) $A^t = A$ (b) $A^c = A$
(c) $(A^t)^t = -A^t$ (d) None

18. A square matrix is skew-symmetric if:

(a) $A^t = -A$ (b) $A^c = -A$
(c) $(A^t)^t = -A^t$ (d) None

19. The matrix $C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a __ matrix.

(a) Diagonal (b) Scalar

(c) Identity (d) Zero

20. The matrix $A=\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$ is a__matrix.

(a) Diagonal (b) Scalar
(c) Identity (d) Zero

21. The matrix $A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$ is a ____ matrix.

(a) Diagonal (b) Identity
(c) Zero (d) None

22. The scalar matrix and identity matrix are ____ matrices.

(a) Diagonal (b) Rectangular
(c) Zero (d) None

23. Every diagonal matrix is not a ___ matrix.

(a) Scalar (b) Identity
(c) Scalar or identity (d) None

24. If A, B are two matrices and A^t, B^t are their respective transpose, then:

(a) $(AB)^t = B^t A^t$ (b) $(AB)^t = A^t B^t$
(c) $A^t B^t = AB$ (d) None

25. If $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ then the determinant of A is:

(a) ad – bc (b) bc – ad
(c) ad + bc (d) bc + ad

26. A square matrix A is called singular if

(a) $|A| \neq 0$ (b) $|A| = 0$
(c) $A = 0$ (d) $A^t = 0$

27. A square matrix A is called non-singular if:

(a) $|A| = 0$ (b) $A = 0$
(c) $|A| \neq 0$ (d) $A^t = 0$

28. Inverse of identity matrix is ___ matrix.

(a) Identity (b) Zero
(c) Rectangular (d) None

29. $AA^{-1} = A^{-1}A =$ _____

(a) Identity matrix
(b) Rectangular matrix
(c) Zero matrix (d) none

30. $(AB)^{-1} =$ ____

(a) $A^{-1}B^{-1}$ (b) $B^{-1}A^{-1}$
(c) BA (d) AB

31. Additive inverse of $\begin{bmatrix}1&-2\\0&-1\end{bmatrix}$ is ____

(a) $\begin{bmatrix}-1&2\\0&1\end{bmatrix}$ (b) $\begin{bmatrix}1&-2\\0&1\end{bmatrix}$
(c) $\begin{bmatrix}-1&-2\\0&-1\end{bmatrix}$ (d) $\begin{bmatrix}1&-2\\0&-1\end{bmatrix}$

Answer Key

1	b	**2**	c	**3**	a	**4**	c	**5**	d
6	a	**7**	a	**8**	c	**9**	d	**10**	a
11	a	**12**	b	**13**	a	**14**	b	**15**	b
16	a	**17**	a	**18**	a	**19**	a	**20**	b
21	b	**22**	a	**23**	c	**24**	a	**25**	a
26	b	**27**	c	**28**	a	**29**	a	**30**	b
31	a								

2. **Complete the following:**

i. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is called …… matrix.

Null / Zero matrix

ii. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is called ……. Matrix.

Identity /Unit matrix

iii. Additive inverse of $\begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$ is …

$\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$

iv. In matrix multiplication, in general, AB …… BA.

$\neq$

v. Matrix A + B may be found if order of A and B is ……

Same

vi. A matrix is called …. matrix if number of rows and columns are equal.

Square

3. **If** $\begin{bmatrix} a+3 & 4 \\ 6 & b-1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ 6 & 2 \end{bmatrix}$, **then find a and b.**

Ans. $\Rightarrow$ $a + 3 = -3$ ……(I)

$b - 1 = 2$ ……..(II)

From (I) $a = -3 - 3$

$a = -6$

From (II) $b = 2 + 1$

$b = 3$

4. **If** $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -4 \\ -2 & -1 \end{bmatrix}$, **then find the following.**

Ans.

(i) 2A + 3B

$$2A + 3B = 2\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} + 3\begin{bmatrix} 5 & -4 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 15 & -12 \\ -6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+15 & 6-12 \\ 2-6 & 0-3 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & -6 \\ -4 & -3 \end{bmatrix}$$

(ii) $-3A + 2B = -3\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 5 & -4 \\ -2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} -6 & -9 \\ -3 & 0 \end{bmatrix} + \begin{bmatrix} 10 & -8 \\ -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -6+10 & -9-8 \\ -3-4 & 0-2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -17 \\ -7 & -2 \end{bmatrix}$$

(iii) $-3\,(A+2B)$

$$A + 2B = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 5 & -4 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 10 & -8 \\ -4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+10 & 3-8 \\ 1-4 & 0-2 \end{bmatrix} = \begin{bmatrix} 12 & -5 \\ -3 & -2 \end{bmatrix}$$

$$-3(A+2B) = -3\begin{bmatrix} 12 & -5 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} -36 & 15 \\ -9 & 6 \end{bmatrix}$$

(iv) $\frac{2}{3}(2A - 3B)$

$$2A - 3B = 2\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} - 3\begin{bmatrix} 5 & -4 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 15 & -12 \\ -6 & -3 \end{bmatrix}$$

$$=\begin{bmatrix}4-15 & 6+12\\ 2+6 & 0+3\end{bmatrix}$$

$$=\begin{bmatrix}-11 & 18\\ 8 & 3\end{bmatrix}$$

$$\frac{2}{3}(2A-3B)=\frac{2}{3}\begin{bmatrix}-11 & 18\\ 8 & 3\end{bmatrix}$$

$$=\begin{bmatrix}\frac{-22}{3} & \frac{36}{3}\\ \frac{16}{3} & \frac{6}{3}\end{bmatrix}=\begin{bmatrix}\frac{-22}{3} & 12\\ \frac{16}{3} & 2\end{bmatrix}$$

5. Find the value of x, if

$$\begin{bmatrix}2 & 1\\ 3 & -3\end{bmatrix}+x=\begin{bmatrix}4 & -2\\ -1 & -2\end{bmatrix}.$$

Ans. $\begin{bmatrix}2 & 1\\ 3 & -3\end{bmatrix}+x=\begin{bmatrix}4 & -2\\ -1 & -2\end{bmatrix}$

$$x=\begin{bmatrix}4 & -2\\ -1 & -2\end{bmatrix}-\begin{bmatrix}2 & 1\\ 3 & -3\end{bmatrix}$$

$$=\begin{bmatrix}4-2 & -2-1\\ -1-3 & -2+3\end{bmatrix}=\begin{bmatrix}2 & -3\\ -4 & 1\end{bmatrix}$$

6. If $A=\begin{bmatrix}0 & 1\\ 2 & -3\end{bmatrix}$, $B=\begin{bmatrix}-3 & 4\\ 5 & -2\end{bmatrix}$,

then prove that

i) $AB \neq BA$

Ans. $AB \neq BA$

$$AB=\begin{bmatrix}0 & 1\\ 2 & -3\end{bmatrix}\begin{bmatrix}-3 & 4\\ 5 & -2\end{bmatrix}$$

$$=\begin{bmatrix}0(-3)+1(5) & 0(4)+1(-2)\\ 2(-3)+-3(5) & 2(4)+-3(-2)\end{bmatrix}$$

$$=\begin{bmatrix}5 & -2\\ -21 & 14\end{bmatrix}$$

$$BA=\begin{bmatrix}-3 & 4\\ 5 & -2\end{bmatrix}\begin{bmatrix}0 & 1\\ 2 & -3\end{bmatrix}$$

$$=\begin{bmatrix}-3(0)+4(2) & -3(1)+4(-3)\\ 5(0)+-2(2) & 5(1)+-2(-3)\end{bmatrix}$$

$$=\begin{bmatrix}8 & -15\\ -4 & 11\end{bmatrix}$$

$AB \neq BA$

7. If $A=\begin{bmatrix}3 & 2\\ 1 & -1\end{bmatrix}$ **and**

$B=\begin{bmatrix}2 & 4\\ -3 & -5\end{bmatrix}$, **then verify that**

(i) $(AB)^t = B^t A^t$

(ii) $(AB)^{-1} = B^{-1} A^{-1}$

Ans. (i) $(AB)^t = B^t A^t$

L.H.S $= (AB)^t$

$$AB=\begin{bmatrix}3 & 2\\ 1 & -1\end{bmatrix}\begin{bmatrix}2 & 4\\ -3 & -5\end{bmatrix}$$

$$=\begin{bmatrix}3(2)+2(-3) & 3(4)+2(-5)\\ 1(2)+-1(-3) & 1(4)+-1(-5)\end{bmatrix}$$

$$=\begin{bmatrix}6-6 & 12-10\\ 2+3 & 4+5\end{bmatrix}$$

$$=\begin{bmatrix}0 & 2\\ 5 & 9\end{bmatrix}$$

$$(AB)^t=\begin{bmatrix}0 & 5\\ 2 & 9\end{bmatrix}$$

R.H.S $= B^t A^t$

$$A^t=\begin{bmatrix}3 & 1\\ 2 & -1\end{bmatrix}$$

$$B^t=\begin{bmatrix}2 & -3\\ 4 & -5\end{bmatrix}$$

$$B^tA^t=\begin{bmatrix}2 & -3\\ 4 & -5\end{bmatrix}\begin{bmatrix}3 & 1\\ 2 & -1\end{bmatrix}$$

$$=\begin{bmatrix}2(3)+-3(2) & 2(1)+-3(-1)\\ 4(3)+-5(2) & 4(1)+-5(-1)\end{bmatrix}$$

$$=\begin{bmatrix}6-6 & 2+3\\ 12-10 & 4+5\end{bmatrix}$$

$$=\begin{bmatrix}0 & 5\\ 2 & 9\end{bmatrix}$$

L.H.S = R.H.S

Hence: $(AB)^t = B^t A^t$

(ii) $\mathbf{(AB)^{-1} = B^{-1} A^{-1}}$

$$A=\begin{bmatrix}3 & 2\\ 1 & -1\end{bmatrix}, B=\begin{bmatrix}2 & 4\\ -3 & -5\end{bmatrix}$$

L.H.S = $(AB)^{-1}$

$$AB=\begin{bmatrix}3 & 2\\ 1 & -1\end{bmatrix}\begin{bmatrix}2 & 4\\ -3 & -5\end{bmatrix}$$

$$=\begin{bmatrix}3(2)+2(-3) & 3(4)+2(-5)\\ 1(2)+-1(-3) & 1(4)-1(-5)\end{bmatrix}$$

$$=\begin{bmatrix}6-6 & 12-10\\ 2+3 & 4+5\end{bmatrix}$$

$$=\begin{bmatrix}0 & 2\\ 5 & 9\end{bmatrix}$$

$$(AB)^{-1}=\frac{1}{|AB|}\operatorname{Adj}AB$$

$$|AB|=\begin{vmatrix}0 & 2\\ 5 & 9\end{vmatrix}=0(9)-5(2)=-10\neq 0$$

$$(AB)^{-1}=\frac{1}{-10}\begin{bmatrix}9 & -2\\ -5 & 0\end{bmatrix}=\frac{1}{10}\begin{bmatrix}-9 & 2\\ 5 & 0\end{bmatrix}=\begin{bmatrix}\frac{-9}{10} & \frac{1}{5}\\ \frac{1}{2} & 0\end{bmatrix}$$

R.H.S = $B^{-1}A^{-1}$

$$A=\begin{bmatrix}3 & 2\\ 1 & -1\end{bmatrix}$$

$$|A|=3(-1)-1(2)=-3-2=-5\neq 0$$

$$\operatorname{Adj}A=\begin{bmatrix}-1 & -2\\ -1 & 3\end{bmatrix}$$

$$A^{-1}=\frac{1}{|A|}\operatorname{Adj}A=\frac{1}{-5}\begin{bmatrix}-1 & -2\\ -1 & 3\end{bmatrix}$$

$$B=\begin{bmatrix}2 & 4\\ -3 & -5\end{bmatrix}$$

$$|B|=2(-5)-(-3)(4)$$

$$=-10+12=2\neq 0$$

$$B^{-1}=\frac{1}{|B|}\operatorname{Adj}B$$

$$=\frac{1}{2}\begin{bmatrix}-5 & -4\\ 3 & 2\end{bmatrix}$$

$$B^{-1}A^{-1}=\left(-\frac{1}{5}\right)\left(\frac{1}{2}\right)\begin{bmatrix}-5 & -4\\ 3 & 2\end{bmatrix}\begin{bmatrix}-1 & -2\\ -1 & 3\end{bmatrix}$$

$$=\frac{1}{-10}\begin{bmatrix}-5(-1)+-4(-1) & -5(-2)+-4(3)\\ 3(-1)+2(-1) & 3(-2)+2(3)\end{bmatrix}$$

$$=\frac{-1}{10}\begin{bmatrix}5+4 & 10-12\\ -3-2 & -6+6\end{bmatrix}=\frac{1}{-10}\begin{bmatrix}9 & -2\\ -5 & 0\end{bmatrix}$$

$$=\begin{bmatrix}\frac{-9}{10} & \frac{-2}{-10}\\ \frac{-5}{-10} & \frac{0}{-10}\end{bmatrix}$$

$$=\begin{bmatrix}\frac{-9}{10} & \frac{1}{5}\\ \frac{1}{2} & 0\end{bmatrix}$$

L.H.S = R.H.S.

Hence: $(AB)^{-1} = B^{-1}A^{-1}$

Unit 02

REAL AND COMPLEX NUMBERS

Define the following:

Natural Numbers

The numbers 1, 2, 3, ... Which we use for counting certain objects are called natural numbers or positive integers. The set of natural numbers is denoted by N.

i.e. $N = \{1, 2, 3, \ldots\}$

Whole Numbers

If we include 0 in the set of natural number, the resulting set is the set of whole numbers, denoted by W,

i.e. $W = \{0, 1, 2, 3, \ldots\}$

Integers

The set of integers consist of positive integers, 0 and negative integers and is denoted by Z

i.e. $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Rational Numbers

All numbers of the form *p/q* where *p*, *q* are integers and *q* is not zero are called rational numbers. The set of rational numbers is denoted by Q,

i.e. $Q=\left\{\frac{p}{q} \mid p,q\in Z \wedge q\neq 0, (p,q)=1\right\}$ or

$$Q = \left\{x \mid x=\frac{p}{q}, p,q\in Z\wedge q\neq 0\right\}$$

Irrational Numbers

The numbers which cannot be expressed as quotient of integers are called irrational numbers.

The set of irrational numbers is denoted by Q',

i.e., $Q'=\left\{x \mid x\neq\frac{p}{q}, p,q\in Z\wedge q\neq 0\right\}$

For example, the numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, π and e are all irrational numbers.

Decimal form of Rational and Irrational number

a) Rational Numbers

The Decimal representation of rational numbers are of two types terminating and recurring

(i) Terminating Decimal Fractions:

The decimal fraction in which there are finite number of digits in its decimal part is called a terminating decimal fraction.

For example $\frac{2}{5}=0.4$ and $\frac{3}{8}=0.375$.

(ii) Recurring and Non-terminating Decimal Fractions

The decimal fraction (non-terminating) in which some digits are repeated again and again in the same order in its decimal part is called a recurring decimal fraction.

For example $\frac{2}{9}=0.2222\ldots$ and $\frac{4}{11}=0.363636\ldots$

b) Irrational Numbers

The decimal representations for irrational numbers are neither terminating nor repeating in blocks. The decimal form

of an irrational number would continue forever and never begin to repeat the same block of digits e.g., $\sqrt{2} = 1.414213562$ ..,

Real Number

The Union of the set of rational numbers and irrational numbers is known as the set of real numbers it is deducted by R.

$$R = Q \cup Q'$$

Hence Q and Q′ are both subsets of R and

$$Q \cap Q' = \phi$$

Example

Express the following decimals in the form $\frac{p}{q}$, where $p, q \in Z$ and $q \neq 0$

(a) $0.\bar{3} = 0.333$ ….

(b) $0.\overline{23} = 0.232323$

Solution

(a) Let $x = 0.\bar{3}$, which can be rewritten as

$x = 0.3333….$ (i)

We multiply both sides of (i) by 10, and obtain

$10\,x = (0.3333\ …) \times 10$

or $10\,x = 3.3333…..$ (ii)

Subtracting (i) from (ii), we have

$10x - x = (3.3333…) - (0.333….)$

or $9x = 3.0000 \Rightarrow x = \frac{1}{3}$

Hence $0.\bar{3} = \frac{1}{3}$

(b) Let $x = 0.\overline{23} = 0.23232323$ …..

We multiply both sides of (i) by 100.

Then $100\,x = (0.232323…….) \times 100$

$100\,x = 23.232323…………$(ii)

Subtracting (i) form (ii), we get

$100x - x = (23.232323….) - (0.232323…)$

$99\,x = 23$

$x = \frac{23}{99}$

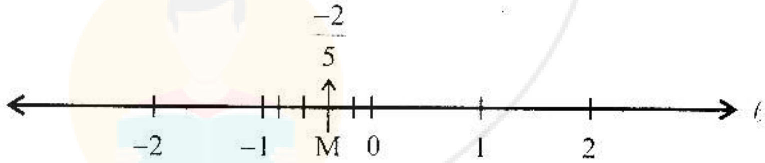
$\Rightarrow$ Thus $0.\overline{23} = \frac{23}{99}$ is a rational number.

Example

Represent the following numbers on the number line.

(i) $\frac{-2}{5}$ (ii) $\frac{15}{7}$

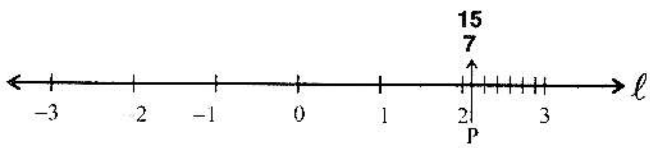
(iii) $-1\frac{7}{9}$



Solution

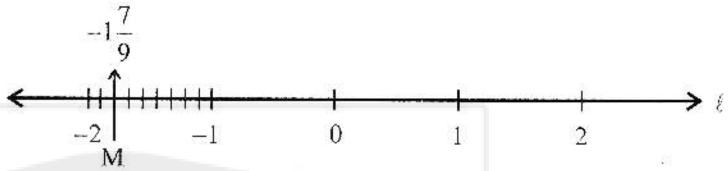
(i) For representing the rational number $-\frac{2}{5}$, on the number line ℓ, divide the unit length between 0 and −1 into five equal parts and take the end of the second part from 0 to its left side. The point M in the following figure represents the rational number $\frac{-2}{5}$.

(ii) $\frac{15}{7} = 2 + \frac{1}{7}$. It lies between 2 and 3.

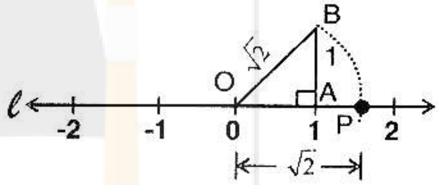


The point P represents the point $\frac{15}{7} = 2\frac{1}{7}$.

(iii) For representing the rational number, $-1\frac{7}{9}$, divide the unit length between -1 and -2 into nine equal parts. Take the end of the 7^{th} part from -1. The point M in the following figure represents the rational number, $-1\frac{7}{9}$.



(iv) Irrational number such as $\sqrt{2}$ can be located on the line ℓ by geometric construction the point corresponding to $\sqrt{2}$ may be constructed by forming a right ΔAOB with sides each of length 1 as shown in the figure.



By Pythagoras theorem, OB = $\sqrt{(1)^2 + (1)^2} = \sqrt{2}$

By drawing an arc with centre at O and radius OB = $\sqrt{2}$ we get point P representing $\sqrt{2}$ on the number line.

Exercise 2.1

Q1. Identify which of the following are relational and irrational numbers.

(i) $\sqrt{3}$ Irrational Number

(ii) $\frac{1}{6}$ Rational Number

(iii) π Irrational Number

(iv) $\frac{15}{2}$ Rational Number

(v) 7.25 Rational Number

(vi) $\sqrt{29}$ Irrational Number

Q2. Convert the following fractions into decimal fraction.

(i) $\frac{17}{25}$

Sol: $\frac{17}{25} = 0.68$

(ii) $\frac{19}{4}$

Sol: $\frac{19}{4} = 4.75$

(iii) $\frac{57}{8}$

Sol: $\frac{57}{8} = 7.125$

(iv) $\frac{205}{18}$

Sol: $\frac{205}{18} = 11.3889$

(v) $\frac{5}{8}$

Sol: $\frac{5}{8} = 0.625$

(vi) $\frac{25}{38}$

Sol: $\frac{25}{38} = 0.65789$

Q2. Which of the following statements are true and which are false?

(i) $\frac{2}{3}$ is an irrational number. False

(ii) π is an irrational number. True

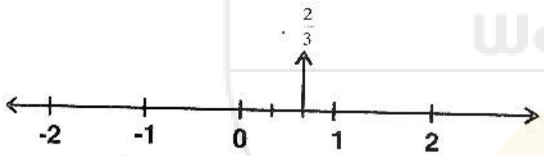
(iii) $\frac{1}{9}$ is a terminating fraction. False

(iv) $\frac{3}{4}$ is a terminating fraction. True

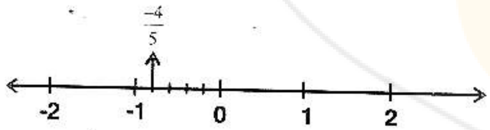
(v) $\frac{4}{5}$ is a recurring fraction. False

Q4. Represent the following numbers on the number line.

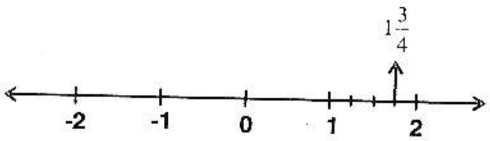
(i) $\frac{2}{3}$



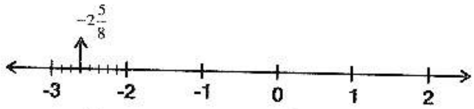
(ii) $-\frac{4}{5}$



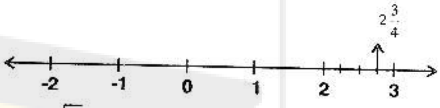
(iii) $1\frac{3}{4}$



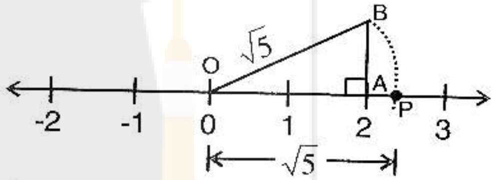
(iv) $-2\frac{5}{8}$



(v) $2\frac{3}{4}$



(vi) $\sqrt{5}$



By Pythagoras theorem

$$OB = \sqrt{(2)^2 + (1)^2} = \sqrt{4+1} = \sqrt{5}$$

By drawing an arc with centre at O and radius OB = $\sqrt{5}$ we get point P representing $\sqrt{5}$ on the number line.

Q5. Give a rational number between $\frac{3}{4}$ and $\frac{5}{9}$.

Ans. The required rational number is the mean of two given numbers, so the required number

$$= \frac{\frac{3}{4} + \frac{5}{9}}{2}$$

$$= \frac{1}{2}\left(\frac{3}{4} + \frac{5}{9}\right)$$

$$= \frac{1}{2}\left(\frac{27+20}{36}\right)$$

$$= \frac{47}{72}$$

Q6. Express the following recurring decimals as the rational number $\frac{p}{q}$, where p, q are integers and q≠0

(i) $0.\overline{5}$

Sol: Let $x = 0.\overline{5}$

$x = 0.55555..........$ (i)

Multiplying both sides by 10

$10x = 10(0.5555......)$

$10x = 5.5555........$ (ii)

Subtracting (i) from (ii)

$10x - x = (5.5555...) - (0.5555...)$

$9x = 5$

$x = \frac{5}{9}$

Hence $0.\overline{5} = \frac{5}{9}$

(ii) $0.\overline{13}$

Sol: Let $x = 0.\overline{13}$

$x = 0.13131313.........$ (i)

Multiplying both sides by 100

$100x = 100(0.131313......)$

$100x = 13.131313.........$ (ii)

Subtracting (i) from (ii)

$100x - x = (13.1313.....) - (0.1313...)$

$99x = 13$

$x = \frac{13}{99}$

Hence $0.\overline{13} = \frac{13}{99}$

(iii) $0.\overline{67}$

Let $x = 0.\overline{67}$

$x = 0.676767.........$ (i)

Multiplying both sides by 100

$100x = 100(0.676767.......)$

$100x = 67.676767.......$ (ii)

Subtracting (i) from (ii)

$100x - x = (67.676767....) - (0.676767...)$

$99x = 67$

$x = \frac{67}{99}$

Hence $0.\overline{67} = \frac{67}{99}$

Properties of Real numbers with respect to Addition and Multiplication

a. Properties of real numbers under addition are as follows:

(i) Closure Property

$a + b \in R, \forall a, b \in R$

e.g., if -3 and $5 \in R$

then $-3 + 5 = 2 \in R$

(ii) Commutative Property

$a + b = b + a, \forall a, b \in R$

e.g., if $2, 3 \in R$

then $2 + 3 = 3 + 2$

or $5 = 5$

(iii) Associative Property

$(a + b) + c = a + (b + c), \forall a, b, c \in R$

e.g., if $5, 7, 3 \in R$

then $(5 + 7) + 3 = 5 + (7 + 3)$

or $12 + 3 = 5 + 10$

or $15 = 15$

(iv) Additive Identity

There exists a unique real number 0 called additive identity such that

$a + 0 = a = 0 + a, \quad \forall a \in R$

(v) Additive Inverse

For every $a \in R$, there exists a unique real number $-a$ called the additive inverse of a such that

$a + (-a) = 0 = (-a) + a$

e.g., additive inverse of 3 is -3 since $3 + (-3) = 0 = (-3) + (3)$

b. Properties of real numbers under multiplication are as follows:

(i) Closure Property

$ab \in R$, $\quad \forall a, b \in R$

e.g., if $-3, 5 \in R$

then $(-3)(5) \in R$

or $-15 \in R$

(ii) Commutative Property:

$ab = ba$, $\quad \forall a, b \in R$

e.g., if $\frac{1}{3}, \frac{3}{2} \in R$

then $\left(\frac{1}{3}\right)\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)\left(\frac{1}{3}\right)$

or $\frac{1}{2} = \frac{1}{2}$

(iii) Associative Property:

$(ab)c = a(bc)$, $\forall a, b, c \in R$

e.g., if $2, 3, 5 \in R$

then $\quad (2 \times 3) \times 5 = 2 \times (3 \times 5)$

or $\quad 6 \times 5 = 2 \times 15$

or $\quad 30 = 30$

(iv) Multiplicative Identity:

There exists a unique real number 1, called the multiplicative identity such that

$a.1 = a = 1.a \ \forall \ a \in R$

(v) Multiplicative Inverse

For every non-zero real number, there exists a unique real number a^{-1} or $\frac{1}{a}$, called multiplicative inverse of a, such that

$aa^{-1} = 1 = a^{-1}a$

or $\quad a \times \frac{1}{a} = 1 = \frac{1}{a} \times a$

e.g., $\quad$ if $5 \in R$, then $\frac{1}{5} \in R$ such that

$$5 \times \frac{1}{5} = 1 = \frac{1}{5} \times 5$$

So, 5 and $\frac{1}{5}$ are multiplicative inverse of each other.

(vi) Multiplication is Distributive over Addition and Subtraction

For all $a, b, c \in R$

$a(b + c) = ab + ac$ (Left distributive law)

$(a + b)c = ac + bc$ (Right distributive law)

e.g., if $2, 3, 5 \in R$, then

$2(3 + 5) = 2 \times 3 + 2 \times 5$

or $\quad 2 \times 8 = 6 + 10$

or $\quad 16 = 16$

And for all $a, b, c \in R$

$a(b - c) = ab - ac$ (Left distributive law)

$(a - b)c = ac - bc$ (Right distributive law)

e.g., if $2, 5, 3 \in R$, then

$2(5 - 3) = 2 \times 5 - 2 \times 3$

or $\quad 2 \times 2 = 10 - 6$

or $\quad 4 = 4$

(b) Properties of Equality of Real Numbers:

Properties of equality of real numbers are as follows:

(i) Reflexive Property

$a = a, \ \forall \ a \in R$

(ii) Symmetric Property

If $a = b$, then $b = a$, $\ \forall a, b \in R$

(iii) Transitive Property

If $a = b$ and $b = c$, then $a = c$, $\ \forall$ a, b, c $\in R$

(iv) Additive Property

If $a = b$, then $a + c = b + c$, $\forall \ a, b, c \in R$

(v) Multiplicative Property

If $a = b$, then $ac = bc$, $\forall \ a, b, c \in R$

(vi) Cancellation Property for Addition
If $a+c=b+c$, then $a=b$, $\forall a,b,c \in$ R

(vii) Cancellation property for Multiplication
If $ac = bc$, $c \neq 0$ then $a = b$, $\forall$ a, b, $c \in$ R

(c) Properties of Inequalities of Real numbers

Properties of inequalities of real numbers are as follows:

(i) Trichotomy Property
$\forall$ a, $b \in$ R
$a < b$ or $a = b$ or $a > b$

(ii) Transitive Property
$\forall$ a, b, $c \in$ R
(a) $a < b$ *and* $b < c \Rightarrow a < c$
(b) $a > b$ *and* $b > c \Rightarrow a > c$

(iii) Multiplicative Property
(a) $\forall$ a, b, $c \in$ R and $c > 0$
(i) $a > b \Rightarrow ac > bc$ (ii) $a < b \Rightarrow ac < bc$
(i) $a > b \Rightarrow ca > cb$ (ii) $a < b \Rightarrow ca < cb$
(b) $\forall$ a, b, $c \in$ R and $c < 0$
(i) $a > b \Rightarrow ac < bc$ (ii) $a < b \Rightarrow ac > bc$
(i) $a > b \Rightarrow ca < cb$ (ii) $a < b \Rightarrow ca > cb$

(iv) Multiplicative Inverse Property:
$\forall$ a, $b \in$ R and $a \neq 0$, $b \neq 0$
(a) $a < b \Leftrightarrow \frac{1}{a} > \frac{1}{b}$
(b) $a > b \Leftrightarrow \frac{1}{a} < \frac{1}{b}$

(v) Additive property:
$\forall$ **a, b c $\in$ R**
(a) $a < b \Rightarrow a + c < b + c$
$a < b \Rightarrow c + a < c + b$
(b) $a > b \Rightarrow a + c > b + c$
$a > b \Rightarrow c + a > c + b$

Exercise 2.2

Q1. Identify the property used in the following.

(i) $a + b = b + a$
commutative property w.r.t. addition

(ii) ab(c)=a(bc)
Associative property w.r.t. multiplication

(iii) 7×1=7 Multiplicative Identity

(iv) $x > y$ or $x=y$ or $x<y$
Trichotomy property of inequality

(v) ab = ba
Commutative property w.r.t. multiplication

(vi) $a + c = b + c \Rightarrow a = b$
Cancellation property for addition

(vii) $5 + (-5) = 0$ Additive Inverse

(viii) $7 \times \frac{1}{7} = 1$ Multiplicative inverse

(ix) $a > b \Rightarrow ac > bc (c > o)$
Multiplicative property of inequality

Q2. Fill in the following blanks by stating the properties of real numbers used.

$3x + 3(y - x)$
$= 3x + 3y - 3x$ Distributive property
$= 3x - 3x + 3y$ Commutative property
$= 0 + 3y$ Additive Inverse (3x, -3x)
$= 3y$ Additive Identity (o+a=a)

Q3. Give the name of property used in the following.

(i) $\sqrt{24} + 0 = \sqrt{24}$ Additive Identity

(ii) $-\frac{2}{3}\left(5+\frac{7}{2}\right)=\left(-\frac{2}{3}\right)(5)+\left(-\frac{2}{3}\right)\left(\frac{7}{2}\right)$

Distributive property of multiplication over addition

(iii) $\pi+(-\pi)=0$ Additive Inverse

(iv) $\sqrt{3}.\sqrt{3}$ **is a real number**

Closure property w.r.t. multiplication

(v) $\left(-\frac{5}{8}\right)\left(-\frac{8}{5}\right)=1$, Multiplicative inverse

Example

Write each radical expression in exponential notation and each exponential expression in radical notation. Do not simplify.

(i) $\sqrt[5]{-8}$ (ii) $\sqrt[3]{x^5}$

(iii) $y^{3/4}$ (iv) $x^{-3/2}$

Solution:

(i) $\sqrt[5]{-8} = (-8)^{1/5}$

(ii) $\sqrt[3]{x^5}=x^{5/3}$

(iii) $y^{3/4}=\sqrt[4]{y^3}$ or $\left(\sqrt[4]{y}\right)^3$

(iv) $x^{-3/2}=\sqrt{x^{-3}}$ or $\left(\sqrt{x}\right)^{-3}$

Example

Simplify $\sqrt[3]{16x^4y^5}$

Solution:

$\sqrt[3]{16x^4y^5} = \sqrt[3]{(2)(8)(x)(x^3)(y^2)(y^3)},$

$= \sqrt[3]{2xy^2(2^3)(x^3)(y^3)}$

$= \sqrt[3]{2xy^2}\,\sqrt[3]{(2^3)(x^3)(y^3)},$

$= \sqrt[3]{2xy^2}\,\sqrt[3]{(2^3)}\,\sqrt[3]{(x^3)}\,\sqrt[3]{(y^3)} = 2xy\sqrt[3]{2xy^2}$

Exercise 2.3

Q1. Write each radical expression in exponential notation and each exponential expression in radical notation. Do not simplify.

(i) $\sqrt[3]{-64}=(-64)^{1/3}$

(ii) $2^{3/5}=\left(2^3\right)^{1/5}=\sqrt[5]{2^3}$

(iii) $-7^{1/3}=-\sqrt[3]{7}$

(iv) $y^{-2/3}=\left(y^{-2}\right)^{1/3}=\sqrt[3]{y^{-2}}$

Q2. Tell whether the following statements are true or false?

(i) $5^{1/5}=\sqrt{5}$ **False**

(ii) $2^{2/3}=\sqrt[3]{4}$ **True**

(iii) $\sqrt{49}=\sqrt{7}$ **False**

(iv) $\sqrt[3]{x^{27}}=x^3$ **False**

Q3. Simplify the following radical expressions.

(i) $\sqrt[3]{-125}=(-125)^{1/3}$

$=\left[(-5)^3\right]^{1/3}=(-5)^{3\times\frac{1}{3}}$

$=-5$

(ii) $\sqrt[4]{32}=\sqrt[4]{16\times 2}$

$=\sqrt[4]{16}\times\sqrt[4]{2}$

$$=\left(2^4\right)^{1/4}\sqrt[4]{2}$$

$$=2^{4\times\frac{1}{4}}\sqrt[4]{2}$$

$$=2\left(\sqrt[4]{2}\right)$$

(iii) $\sqrt[5]{\frac{3}{32}}$

$$=\frac{\sqrt[5]{3}}{\sqrt[5]{32}}$$

$$=\frac{\sqrt[5]{3}}{\left(2^5\right)^{\frac{1}{5}}}$$

$$=\frac{\sqrt[5]{3}}{2^{\cancel{5}\times\frac{1}{\cancel{5}}}}$$

$$=\frac{\sqrt[5]{3}}{2}$$

(iv) $\sqrt[3]{\frac{-8}{27}}$

$$=\left(\frac{-8}{27}\right)^{\frac{1}{3}}$$

$$=\left[\left(\frac{-2}{3}\right)^3\right]^{\frac{1}{3}}$$

$$=\left(\frac{-2}{3}\right)^{3\times\frac{1}{3}} \qquad =\frac{-2}{3}$$

Example

Use rules of exponents to simplify each expression and write the answer in terms of positive exponents.

(i) $$\frac{x^{-2}x^{-3}y^7}{x^{-3}y^4}=\frac{x^{-5}y^7}{x^{-3}y^4}=x^{-5+3}y^{7-4}=x^{-2}y^3=\frac{y^3}{x^2}$$

(ii) $$\left(\frac{4a^3b^o}{9a^{-5}}\right)^{-2}=\left(\frac{4a^{3+5}\ x\,1}{9}\right)^{-2}=\left(\frac{4a^8}{9}\right)^{-2}=\left(\frac{9}{4a^8}\right)^2=\frac{81}{16a^{16}}$$

Example

Simplify the following by using laws of indices:

(i) $\left(\frac{8}{125}\right)^{-4/3}$ (ii) $\frac{4(3)^n}{3^{n+1}-3^n}$

Solution

Using Laws of Indices.

(i) $$\left(\frac{8}{125}\right)^{-4/3} = \left(\frac{125}{8}\right)^{4/3}=\frac{(125)^{4/3}}{(8)^{4/3}}=\frac{(5^3)^{4/3}}{(2^3)^{4/3}}=\frac{5^4}{2^4}=\frac{625}{16}$$

(ii) $$\frac{4(3)^n}{3^{n+1}-3^n} = \frac{4(3)^n}{3^n[3-1]}=\frac{4(3)^n}{2(3^n)}=\frac{4}{2}=2$$

Exercise 2.4

Q1. Use laws of exponents to simplify

(i) $$\frac{(243)^{-\frac{2}{3}}(32)^{-\frac{1}{5}}}{\sqrt{(196)^{-1}}}$$

$$=\frac{\sqrt{196}}{(243)^{\frac{2}{3}}(32)^{\frac{1}{5}}}$$

$$=\frac{\sqrt{14\times14}}{(3\times3\times3\times3\times3)^{\frac{2}{3}}(2\times2\times2\times2\times2)^{\frac{1}{5}}}$$

$$=\frac{\sqrt{(14)^2}}{\left(3^3\times3^2\right)^{\frac{2}{3}}\left(2^5\right)^{\frac{1}{5}}}$$

$$=\frac{14}{3^{\cancel{3}\times\frac{2}{\cancel{3}}}\times3^{2\times\frac{2}{3}}\times2^{\cancel{5}\times\frac{1}{\cancel{5}}}}$$

$$=\frac{\cancel{14}7}{3^2\times3^{2\times\frac{2}{3}}\times\cancel{2}}$$

$$=\frac{7}{3^2\times3^{\frac{4}{3}}}$$

$$=\frac{7}{3^2\times3^{\frac{3+1}{3}}}$$

$$=\frac{7}{3^2\times3^{\frac{3}{3}+\frac{1}{3}}}$$

$$=\frac{7}{3^2\times3\times3^{\frac{1}{3}}}$$

$$=\frac{7}{3^3\times\sqrt[3]{3}}$$

$$=\frac{7}{27\left(\sqrt[3]{3}\right)}$$

(ii) $$\left(2x^5y^{-4}\right)\left(-8x^{-3}.y^2\right)$$

$$=2(-8)x^{5-3}.y^{-4+2}$$

$$=-16x^2.y^{-2}$$

$$=-16\frac{x^2}{y^2}$$

(iii) $$\left(\frac{x^{-2}y^{-1}z^{-4}}{x^4y^{-3}z^0}\right)^{-3}$$

$$=\left(x^{-2-4}.y^{-1+3}.z^{-4+0}\right)^{-3}$$

$$=\left(x^{-6}.y^2.z^{-4}\right)^{-3}$$

$$=x^{-6(-3)}.y^{2(-3)}.z^{-4(-3)}$$

$$=x^{18}.y^{-6}.z^{12}$$

$$=\frac{x^{18}.z^{12}}{y^6}$$

(iv) $$\frac{(81)^n.3^5-(3)^{4n-1}(243)}{\left(9^{2n}\right)\left(3^3\right)}$$

$$=\frac{\left(3^4\right)^n.3^5-(3)^{4n-1}\left(3^5\right)}{\left(3^2\right)^{2n}\left(3^3\right)}$$

$$=\frac{3^{4n+5}-3^{4n-1+5}}{3^{4n+3}}$$

$$=\frac{3^{4n+3+2}-3^{4n+4}}{3^{4n+3}}$$

$$=\frac{3^{4n+3+2}-3^{4n+3+1}}{3^{4n+3}}$$

$$=\frac{3^{4n+3}.3^2-3^{4n+3}.3^1}{3^{4n+3}}$$

$$=\frac{\cancel{3^{4n+3}}\left(3^2-3^1\right)}{\cancel{3^{4n+3}}}$$

$$=9-3$$

$$=6$$

Q2. Show that

$$\left(\frac{x^a}{x^b}\right)^{a+b}\times\left(\frac{x^b}{x^c}\right)^{b+c}\times\left(\frac{x^c}{x^a}\right)^{c+a}=1$$

Sol: L.H.S

$$=\left(\frac{x^a}{x^b}\right)^{a+b}\times\left(\frac{x^b}{x^c}\right)^{b+c}\times\left(\frac{x^c}{x^a}\right)^{c+a}$$

$$=\left(x^{a-b}\right)^{a+b}\times\left(x^{b-c}\right)^{b+c}\times\left(x^{c-a}\right)^{c+a}$$

$$=x^{(a-b)(a+b)}\times x^{(b-c)(b+c)}\times x^{(c-a)(c+a)}$$

$$=x^{a^2-b^2}\times x^{b^2-c^2}\times x^{c^2-a^2}$$

$$=x^{\cancel{a^2}-\cancel{b^2}+\cancel{b^2}-\cancel{c^2}+\cancel{c^2}-\cancel{a^2}}$$

$$=x^0$$

$$=1$$

= R.H.S

Q3. Simplify

(i) $$\frac{2^{1/3}\times(27)^{1/3}\times(60)^{1/2}}{(180)^{1/2}\times(4)^{-1/3}\times(9)^{1/4}}$$

$$=\frac{2^{1/3}\times\left(3^3\right)^{1/3}\times\left(2^2\times3\times5\right)^{1/2}}{\left(2^2\times3^2\times5\right)^{1/2}\times\left(2^2\right)^{-1/3}\times\left(3^2\right)^{1/4}}$$

$$=\frac{2^{\frac{1}{3}}\times3^{3\times\frac{1}{3}}\times2^{2\times\frac{1}{2}}\times3^{\frac{1}{2}}\times5^{\frac{1}{2}}}{2^{2\times\frac{1}{2}}\times3^{2\times\frac{1}{2}}\times5^{\frac{1}{2}}\times2^{2\left(-\frac{1}{3}\right)}\times3^{2\times\frac{1}{4}}}$$

$$=\frac{2^{\frac{1}{3}}\times3\times2\times3^{\frac{1}{2}}\times5^{\frac{1}{2}}}{2\times3\times5^{\frac{1}{2}}\times2^{\frac{-2}{3}}\times3^{\frac{1}{2}}}$$

$$=2^{\frac{1}{3}+1-1+\frac{2}{3}}\times3^{1+\frac{1}{2}-1-\frac{1}{2}}\times5^{\frac{1}{2}-\frac{1}{2}}$$

$$=2^{\frac{3}{3}}\times3^{o}\times5^{o}$$

$$=2\times1\times1$$

$$=2$$

(ii) $$\sqrt{\frac{(216)^{\frac{2}{3}}\times(25)^{\frac{1}{2}}}{(.04)^{-\frac{1}{2}}}}$$

$$=\sqrt{\frac{\left(6^3\right)^{\frac{2}{3}}\times\left(5^2\right)^{\frac{1}{2}}}{\left(\frac{4}{100}\right)^{-\frac{1}{2}}}}$$

$$=\sqrt{\frac{6^{\cancel{3}\times\frac{2}{\cancel{3}}}\times5^{\cancel{2}\times\frac{1}{\cancel{2}}}}{\left(\frac{100}{4}\right)^{\frac{1}{2}}}}\qquad=\sqrt{\frac{6^2\times5}{(25)^{\frac{1}{2}}}}$$

$$=\sqrt{\frac{6^2\times5}{5^{\cancel{2}\times\frac{1}{\cancel{2}}}}}$$

$$=\sqrt{\frac{6^2\times\cancel{5}}{\cancel{5}}}$$

$$=\sqrt{6^2}$$

$$=6$$

(iii) $$5^{2^3}\div\left(5^2\right)^3$$

$$=5^8\div5^6$$

$$=\frac{5^8}{5^6}$$

$$=5^{8-6}$$

$$=5^2$$

$$=25$$

iv) $\left(x^3\right)^2 \div x^{3^2}$

$= x^6 \div x^9$

$= \dfrac{x^6}{x^9}$

$= \dfrac{1}{x^{9-6}} = \dfrac{1}{x^3}$

Definition of a Complex Number

A number of the form $z = a + bi$ where a and b are real numbers and $i = \sqrt{-1}$, is called a complex number and is represented by z i.e., $z = a + ib$

Set of Complex Numbers

The set of all complex numbers is denoted by C and

C = { $z \mid z = a + bi$, where $a, b \in$ R and $i = \sqrt{-1}$ }

The numbers a and b, called the real and imaginary parts of z, are denoted as $a = \text{Re}(z)$ and $b = \text{Im}(z)$ respectively.

Conjugate of a Complex Number

If we change i to $-i$ in $z = a + bi$, we obtain another complex number $a - bi$ called the complex conjugate of z and is denoted by $\bar{z}$ (read z bar).

Thus, if $z = -1 - i$, *then* $\bar{z} = -1 + i$.

The number a + bi and a − bi are called conjugates of each other.

Equality of Complex Numbers and its Properties

For all $a, b, c, d \in$ R,

$a+bi=c+di$ if and only if $a = c$ and $b = d$.

e.g., $2x + y^2 i = 4 + 9i$

if and only if

$2x = 4$ and $y^2 = 9$, i.e., $x = 2$ and $y = \pm 3$

Properties of real numbers R are also valid for the set of complex numbers.

(i) $Z_1 = Z_2$, (Reflexive Law)

(ii) If $Z_1 = Z_2$, then $Z_2 = Z_1$ (Symmetric law)

(iii) If $Z_1 = Z_2$, and $Z_2 = Z_3$ then $Z_1 = Z_3$ (transitive law)

Exercise 2.5

Q1. Evaluate

(i) i^7

$= i^6 . i$

$= \left(i^2\right)^3 . i$

$= (-1)^3 . i$

$= -1 . i$

$= -i$

(ii) i^{50}

$= \left(i^2\right)^{25}$

$= (-1)^{25}$

$= -1$

(iii) i^{12}

$= \left(i^2\right)^6$

$= (-1)^6$

$= 1$

(iv) $(-i)^8$

$= i^8$

$= \left(i^2\right)^4$

$= (-1)^4$

$= 1$

(v) $(-i)^5$

$= -i^5$

$= -\left(i^4.i\right)$

$= -\left(\left(i^2\right)^2.i\right)$

$= -\left((-1)^2.i\right)$

$-(i)$

$= -i$

(vi) i^{27}

$= i^{26}.\, i$

$= \left(i^2\right)^{13}.\, i$

$= (-1)^{13}.\, i$

$= (-1)i$

$= -i$

Q2. Write the conjugate of the following numbers.

(i) $2+3i$

Let $z = 2+3i$

then $\bar{z} = 2-3i$

(ii) $3-5i$

Let $z = 3-5i$

$\bar{z} = 3+5i$

(iii) $-i$

Sol: Let $z = 0-i$

then $\bar{z} = 0+i = i$

(iv) $-3+4i$

Let $z = -3+4i$

then $\bar{z} = -3-4i$

(v) $-4-i$

Let $z = -4-i$

then $\bar{z} = -4+i$

vi) $i-3$

Let $z = -3+i$

then $\bar{z} = -3-i$

Q3. Write the real and imaginar part of the following numbe

(i) $1+i$

Let $z = 1+i$

Re $(z) = 1$, Im $(z) = 1$

(ii) $-1+2i$

Let $z = -1+2i$

Re $(z) = -1$, Im $(z) = 2$

(iii) $-3i+2$

Let $z = 2-3i$

Re $(z) = 2$, Im $(z) = -3$

(iv) $-2-2i$

Let $z = -2-2i$

Re $(z) = -2$, Im $(z) = -2$

(v) $-3i$

Let $z = 0-3i$

Re $(z) = 0$, Im $(z) = -3$

(vi) $2+0i$

Let $z = 2+0i$

Re $(z) = 2$, Im $(z) = 0$

Q4. Find the value of x and y if

$x+iy+1 = 4-3i$

Sol: $x+iy+1 = 4-3i$

$x+iy = 4-1-3i$

$x+iy = 3-3i$

Two complex numbers are equal if the real and imaginary parts are equal

So $x = 3$ and $y = -3$

Basic Operations on Complex Numbers

(i) **Addition:**

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers and $a, b, c, d \in R$.

The sum of two complex numbers is given by

$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i$

i.e., the sum of two complex numbers is the sum of the corresponding real and the imaginary parts.

e.g., $(3 - 8i) + (5 + 2i) = (3 + 5) + (-8 + 2)i = 8 - 6i$

(ii) **Multiplication:**

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers and $a, b, c, d \in R$.

The products are found as

(i) If $k \in R$, $kz_1 = k(a + bi) = ka + kbi$.

(Multiplication of a complex number with a scalar)

(ii) $Z_1 Z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$

(Multiplication of two complex numbers)

The multiplication of any two complex numbers $(a + bi)$ and $(c + di)$ is explained as

$$\begin{aligned} z_1z_2 &= (a + bi)(c + di) = a(c + di) + bi(c + di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci + bd(-1) && \text{(since } i^2 = -1\text{)} \\ &= (ac - bd) + (ad + bc)i && \text{(combining like terms)} \end{aligned}$$

e.g., $(2 - 3i)(4 + 5i) = 8 + 10i - 12i - 15i^2 = 23 - 2i.$ (since $i^2 = -1$)

(iii) **Subtraction:**

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers and $a, b, c, d \in R$.

The difference between two complex numbers is given by

$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i$

e.g.., $(-2 + 3i) - (2 + i) = (-2 - 2) + (3 - 1)i = -4 + 2i$

i.e., the difference of two complex numbers is the difference of the corresponding real and imaginary parts.

iv) **Division:**

Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers and $a, b, c, d, \in R$.

The division of $a + bi$ by $c + di$ is given by

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \frac{a+bi}{c+di} \times \frac{c-di}{c-di}$$

(Multiplying the numerator and denominator by $c-di$, the complex conjugate of $c+di$).

$$= \frac{ac+bci-adi-bdi^2}{c^2-(di)^2}$$

$$= \frac{ac+bci-adi+bd}{c^2+d^2}, \text{since } i^2=-1$$

$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \left(\frac{bc-ad}{c^2+d^2}\right)i$$

Example

Separate the real and imaginary parts of $(-1+\sqrt{-2})^2$

Solution

Let $z = -1+\sqrt{-2}$, then

$z^2 = (-1+\sqrt{-2})^2 = (-1+i\sqrt{2})^2$, changing to i-form

$= (-1+i\sqrt{2})(-1+i\sqrt{2}) = (-1)(-1+i\sqrt{2}) + i\sqrt{2}(-1+i\sqrt{2})$

$= 1 - i\sqrt{2} - i\sqrt{2} + 2i^2 = -1 - 2\sqrt{2}\,i$

Hence $Re(z^2) = -1$ and $Im(z^2) = -2\sqrt{2}$

Example

Express $\frac{1}{1+2i}$ in the standard form $a+bi$.

Solution

We have $\frac{1}{1+2i} = \frac{1}{1+2i} \times \frac{1-2i}{1-2i}$

(multiplying the numerator and dominator by $\overline{1+2i}$

$= \frac{1-2i}{1-(2i)^2} = \frac{1-2i}{1-4i^2}$, (simplifying)

$= \frac{1-2i}{5}$, (since $i^2=-1$)

$= \frac{1}{5} - \frac{2}{5}i$, which is of the form $a+bi$

Example

Express $\frac{4+5i}{4-5i}$ in the standard form $a+bi$.

Solution

$\frac{4+5i}{4-5i} = (4+5i) \cdot \frac{1}{4-5i} \times \frac{4+5i}{4+5i}$

(multiplying and dividing by the conjugate of $(4-5i)$)

$= \frac{(4+5i)^2}{(4)^2-(5i)^2} = \frac{16+40i+25i^2}{16-25i^2}$ (simplifying)

$= \frac{16+40i-25}{16+25}$ (since $i^2=-1$)

$= \frac{-9+40i}{41} = \frac{-9}{41} + \frac{40}{41}i$

Example

Solve $(3-4i)(x+yi) = 1+0.i$ for real numbers x and y, where $i=\sqrt{-1}$.

Solution

We have $(3-4i)(x+yi) = 1+0.i$

or $3x+3iy-4ix-4i^2y = 1+0.i$

or $3x+3iy-4ix-4(-1)y = 1+0.i$

or $3x+4y+(3y-4x)i = 1+0.i$

Equating the real and imaginary parts, we obtain

$3x+4y=1$ and $3y-4x=0$

Solving these two equations simultaneously, we have $x=\frac{3}{25}$ and $y=\frac{4}{25}$

Exercise 2.6

Q1. Identify the following statements as true or false.

(i) $\sqrt{-3}\times\sqrt{-3}=3$ False

(ii) $i^{73}=-i$ False

(iii) $i^{10}=-1$ True

(iv) Complex conjugate of $\left(-6i+i^2\right)$ is $(-1+6i)$ True

(v) Difference of a complex number $z=a+bi$ and its conjugate is a real number. False

(vi) If $(a-1)-(b+3)i=5+8i$ then $a=6$ and $b=-11$. True

(vii) Product of a complex number and its conjugate is always a non-negative real number. True

Q2. Express each complex number in the standard form $a+bi$, where 'a' and 'b' are real numbers.

(i) $(2+3i)+(7-2i)$

$=2+3i+7-2i$

$=(2+7)+(3-2)i$

$=9+i$

(ii) $2(5+4i)-3(7+4i)$

$=10+8i-21-12i$

$=(10-21)+(8-12)i$

$=-11-4i$

(iii) $-1(-3+5i)-(4+9i)$

$=3-5i-4-9i$

$=(3-4)+(-5-9)i$

$=-1-14i$

(iv) $2i^2+6i^3+3i^{16}-6i^{19}+4i^{25}$

$=2(-1)+6i^2.i+3\left(i^2\right)^8-6i^{18}.i+4i^{24}.i$

$=-2+6(-1).i+3(-1)^8-6\left(i^2\right)^9.i+4\left(i^2\right)^{12}.i$

$=-2-6i+3(1)-6(-1)^9.i+4(-1)^{12}.i$

$=-2-6i+3-6(-1)i+4(1).i$

$=-2-\cancel{6i}+3+\cancel{6i}+4i$

$=1+4i$

Q3. Simplify and write your answer in the form $a+bi$

(i) $(-7+3i)(-3+2i)$

$=21-14i-9i+6i^2$

$=21-23i+6(-1)$

$=21-6-23i$

$=15-23i$

(ii) $\left(2-\sqrt{-4}\right)\left(3-\sqrt{-4}\right)$

$=\left(2-\sqrt{4}.\sqrt{-1}\right)\left(3-\sqrt{4}\sqrt{-1}\right)$

$=(2-2i)(3-2i)$

$=6-4i-6i+4i^2$

$=6-10i+4(-1)$

$=6-10i-4$

$=2-10i$

(iii) $\left(\sqrt{5}-3i\right)^2$

$=\left(\sqrt{5}\right)^2+(3i)^2-2\left(\sqrt{5}\right)(3i)$

$=5+9i^2-6\sqrt{5}i$

$=5+9(-1)-6\sqrt{5}i$

$=5-9-6\sqrt{5}i$

$=-4-6\sqrt{5}i$

(iv) $(2-3i)\left(\overline{3-2i}\right)$

$=(2-3i)(3+2i)$

$=6+4i-9i-6i^2$

$=6-5i-6(-1)$

$=6-5i+6$

$=12-5i$

Q4. Simplify and write your answer in the form of $a+bi$

(i) $\dfrac{-2}{1+i}$

$=\dfrac{-2}{1+i}\times\dfrac{1-i}{1-i}$

$=\dfrac{-2(1-i)}{(1)^2-(i)^2}$

$=\dfrac{-2(1-i)}{1-i^2}$

$=\dfrac{-2(1-i)}{1-(-1)}$

$=\dfrac{-2(1-i)}{1+1}$

$=\dfrac{-\cancel{2}(1-i)}{\cancel{2}}$

$=-(1-i)$

$=-1+i$

(ii) $\dfrac{2+3i}{4-i}$

$=\dfrac{2+3i}{4-i}\times\dfrac{4+i}{4+i}$

$=\dfrac{(2+3i)(4+i)}{(4)^2-(i)^2}$

$=\dfrac{8+2i+12i+3i^2}{16-i^2}$

$=\dfrac{8+14i+3(-1)}{16-(-1)}$

$=\dfrac{8+14i-3}{16+1}$

$=\dfrac{5+14i}{17}$

$=\dfrac{5}{17}+\dfrac{14}{17}i$

(iii) $\dfrac{9-7i}{3+i}$

$=\dfrac{9-7i}{3+i}\times\dfrac{3-i}{3-i}$

$=\dfrac{(9-7i)(3-i)}{(3)^2-(i)^2}$

$=\dfrac{27-9i-21i+7i^2}{9-i^2}=\dfrac{27-30i+7(-1)}{9-(-1)}$

$=\dfrac{27-7-30i}{9+1}$

$=\dfrac{20-30i}{10}$

$=\dfrac{20}{10}-\dfrac{30}{10}i$

$=2-3i$

(iv) $\dfrac{2-6i}{3+i}-\dfrac{4+i}{3+i}$

$=\dfrac{(2-6i)-(4+i)}{3+i}$

$=\dfrac{2-6i-4-i}{3+i}$

$=\dfrac{-2-7i}{3+i}$

$=\dfrac{-2-7i}{3+i}\times\dfrac{3-i}{3-i}$

$$=\frac{(-2-7i)(3-i)}{(3)^2-(i)^2}$$

$$=\frac{-6+2i-21i+7i^2}{9-i^2}$$

$$=\frac{-6-19i+7(-1)}{9-(-1)}$$

$$=\frac{-6-7-19i}{9+1}$$

$$=\frac{-13-19i}{10}$$

$$=\frac{-13}{10}-\frac{19}{10}i$$

) $\left(\frac{1+i}{1-i}\right)^2$

$$=\frac{(1)^2+(i)^2+2(1)(i)}{(1)^2+(i)^2-2(1)(i)}$$

$$=\frac{1+i^2+2i}{1+i^2-2i}$$

$$=\frac{\cancel{1}-\cancel{1}+2i}{\cancel{1}-\cancel{1}-2i}$$

$$=\frac{2i}{-2i}$$

$$=-1$$

$$=-1+0\,i$$

$$\frac{1}{(2+3i)(1-i)}$$

$$=\frac{1}{2-2i+3i-3i^2}$$

$$=\frac{1}{2+i-3(-1)}$$

$$=\frac{1}{2+i+3}$$

$$=\frac{1}{5+i}$$

$$=\frac{1}{5+i}\times\frac{5-i}{5-i}$$

$$=\frac{5-i}{(5)^2-(i)^2}$$

$$=\frac{5-i}{25-i^2}$$

$$=\frac{5-i}{25-(-1)}$$

$$=\frac{5-i}{25+1}$$

$$=\frac{5-i}{26}$$

$$=\frac{5}{26}-\frac{1}{26}i$$

Q5. Calculate (a) $\bar{z}$ **(b)** $z+\bar{z}$ **c)** $z-\bar{z}$ **(d)** $z.\bar{z}$ **for each of the following.**

(i) $z=0-i$

(a) $\bar{z}=0+i$

(b) $z+\bar{z}=0-i+0+i=0$

(c) $z-\bar{z}=0-i-(0+i)$

$=0-i-0-i$

$=-2\,i$

(d) $z.\,\bar{z}=(0-i)(0+i)$

$=(0)^2-(i)^2=0-(-1)$

$=1$

(ii) $z=2+i$

(a) $\bar{z}=2-i$

(b) $z+\bar{z}=2+\cancel{i}+2-\cancel{i}$

$=4$

(c) $z-\bar{z}=(2+i)-(2-i)$

$=\cancel{2}+i-\cancel{2}+i$

$=2i$

(d) $z.\bar{z}=(2+i)(2-i)$

$=(2)^2-(i)^2$

$=4-i^2$

$=4-(-1)$

$=4+1$

$=5$

(iii) $z=\dfrac{1+i}{1-i}$

$=\dfrac{1+i}{1-i}\times\dfrac{1+i}{1+i}$

$=\dfrac{(1+i)^2}{(1)^2-(i)^2}$

$=\dfrac{(1)^2+(i)^2+2(1)(i)}{1-i^2}$

$=\dfrac{1+i^2+2i}{1-(-1)}=\dfrac{1-1+2i}{1+1}$

$=\dfrac{\cancel{2}i}{\cancel{2}}=i$

$z=0+i$

(a) $\bar{z}=0-i$

(b) $z+\bar{z}=0+i+0-i=0$

(c) $z-\bar{z}=0+i-(0-i)$

$=0+i-0+i$

$=2i$

(d) $z.\bar{z}=(0+i)(0-i)$

$=(0)^2-(i)^2=0-(-1)$

$=0+1=1$

(iv) $z=\dfrac{4-3i}{2+4i}$

$=\dfrac{4-3i}{2+4i}\times\dfrac{2-4i}{2-4i}$

$=\dfrac{(4-3i)(2-4i)}{(2)^2-(4i)^2}$

$=\dfrac{8-16i-6i+12i^2}{4-16i^2}$

$=\dfrac{8-22i+12(-1)}{4-16(-1)}$

$=\dfrac{8-12-22i}{4+16}$

$=\dfrac{-4-22i}{20}$

$=-\dfrac{4}{20}-\dfrac{22}{20}i$

$z=-\dfrac{1}{5}-\dfrac{11}{10}i$

(a) $\bar{z}=-\dfrac{1}{5}+\dfrac{11}{10}i$

(b) $z+\bar{z}=-\dfrac{1}{5}+\cancel{\dfrac{11}{10}i}-\dfrac{1}{5}+\cancel{\dfrac{11}{10}i}$

$=-\dfrac{2}{5}$

(c) $z-\bar{z}=-\dfrac{1}{5}-\dfrac{11}{10}i-\left(-\dfrac{1}{5}+\dfrac{11}{10}i\right)$

$=-\cancel{\dfrac{1}{5}}-\dfrac{11}{10}i+\cancel{\dfrac{1}{5}}-\dfrac{11}{10}i$

$=-\dfrac{22}{10}i$

$=-\dfrac{11}{5}i$

(d) $z.\bar{z}=\left(-\dfrac{1}{5}-\dfrac{11}{10}i\right)\left(-\dfrac{1}{5}+\dfrac{11}{10}i\right)$

$=\left(-\dfrac{1}{5}\right)^2-\left(\dfrac{11}{10}i\right)^2$

$=\dfrac{1}{25}-\dfrac{121}{100}i^2$

$$=\frac{1}{25}-\frac{121}{100}(-1)$$
$$=\frac{1}{25}+\frac{121}{100}$$
$$=\frac{4+121}{100}$$
$$=\frac{125}{100}$$
$$=\frac{5}{4}$$

Q6. If $z=2+3i$ and $w=5-4i$, show that:

(i) $\overline{z+w}=\bar{z}+\bar{w}$

Sol: L.H.S $=\overline{z+w}$

$$z+w=2+3i+5-4i$$
$$z+w=7-i$$
$$\overline{z+w}=7+i$$

Now R.H.S $=\bar{z}+\bar{w}$

$$\bar{z}=2-3i$$
$$\bar{w}=5+4i$$
$$\bar{z}+\bar{w}=2-3i+5+4i$$
$$=7+i$$

Hence $\overline{z+w}=\bar{z}+\bar{w}$

(ii) $\overline{z-w}=\bar{z}-\bar{w}$

Sol: L.H.S $=\overline{z-w}$

$$z-w=2+3i-(5-4i)$$
$$=2+3i-5+4i$$
$$=-3+7i$$
$$\overline{z-w}=-3-7i$$

R.H.S $=\bar{z}-\bar{w}$

$$\bar{z}=2-3i$$
$$\bar{w}=5+4i$$
$$\bar{z}-\bar{w}=(2-3i)-(5+4i)$$
$$=2-3i-5-4i$$
$$=-3-7i$$

Hence $\overline{z-w}=\bar{z}-\bar{w}$

(iii) $\overline{z.w}=\bar{z}.\bar{w}$

L.H.S $=\overline{z.w}$

$$z.w=(2+3i)(5-4i)$$
$$=10-8i+15i-12i^2$$
$$=10+7i-12(-1)$$
$$=10+7i+12$$
$$=22+7i$$
$$\overline{z.w}=22-7i$$

R.H.S $=\bar{z}.\bar{w}$

$$\bar{z}=2-3i$$
$$\bar{w}=5+4i$$
$$\bar{z}.\bar{w}=(2-3i)(5+4i)$$
$$=10+8i-15i-12i^2$$
$$=10-7i-12(-1)$$
$$=10-7i+12$$
$$\bar{z}.\bar{w}=22-7i$$

Hence $\overline{z.w}=\bar{z}.\bar{w}$

(iv) $\overline{\left(\frac{z}{w}\right)}=\frac{\bar{z}}{\bar{w}}$, where $w\neq 0$

LHS $=\overline{\left(\frac{z}{w}\right)}$

$$\frac{z}{w}=\frac{2+3i}{5-4i}$$
$$=\frac{2+3i}{5-4i}\times\frac{5+4i}{5+4i}$$
$$=\frac{(2+3i)(5+4i)}{(5)^2-(4i)^2}=\frac{10+8i+15i+12i^2}{25-16i^2}$$

$$=\frac{10+23i+12(-1)}{25-16(-1)}$$

$$=\frac{10-12+23i}{25+16}$$

$$=\frac{-2+23i}{41}$$

$$=-\frac{2}{41}+\frac{23}{41}i$$

$$\overline{\left(\frac{z}{w}\right)}=-\frac{2}{41}-\frac{23}{41}i$$

R.H.S $=\dfrac{\bar{z}}{\bar{w}}$

$$\bar{z}=2-3i$$

$$\bar{w}=5+4i$$

$$\frac{\bar{z}}{\bar{w}}=\frac{2-3i}{5+4i}$$

$$=\frac{(2-3i)(5-4i)}{(5)^2-(4i)^2}$$

$$=\frac{10-8i-15i+12i^2}{25-16i^2}$$

$$=\frac{10-23i+12(-1)}{25-16(-1)}$$

$$=\frac{10-12-23i}{25+16}$$

$$=\frac{-2-23i}{41}$$

$$=-\frac{2}{41}-\frac{23}{41}i$$

Hence $\overline{\left(\dfrac{z}{w}\right)}=\dfrac{\bar{z}}{\bar{w}}$

(v) $\dfrac{1}{2}(z+\bar{z})$ is the real part of z

Sol: $z=2+3i$

Now $\bar{z}=2-3i$

$$\frac{1}{2}(z+\bar{z})=\frac{1}{2}(2+\cancel{3i}+2-\cancel{3i})$$

$$=\frac{1}{\cancel{2}}(\cancel{4})$$

$$\frac{1}{2}(z+\bar{z})=2$$

$$\frac{1}{2}(z+\bar{z})=\text{Re}(z)$$

Hence $\dfrac{1}{2}(z+\bar{z})$ is equal to the real part of z.

(vi) $\dfrac{1}{2i}(z-\bar{z})$ is the real part of z.

Sol. $z=2+3i$

Now $\bar{z}=2-3i$

$$\frac{1}{2i}(z-\bar{z})=\frac{1}{2i}\left[(2+3i)-(2-3i)\right]$$

$$=\frac{1}{2i}(\cancel{2}+3i-\cancel{2}+3i)$$

$$=\frac{6i}{3i}$$

$$=3$$

$$\frac{1}{2i}(z-\bar{z})=\text{R}(z)$$

Hence proved that $\dfrac{1}{2i}(z-\bar{z})$ is equal to the real part of z.

Q7. Solve the following equation for real x and y

(i) $(2-3i)(x+yi)=4+i$

$$(x+yi)=\frac{4+i}{2-3i}$$

$$=\frac{4+i}{2-3i}\times\frac{2+3i}{2+3i}$$

$$=\frac{(4+i)(2+3i)}{(2)^2-(3i)^2}$$

$$=\frac{8+12i+2i+3i^2}{4-9i^2}$$

$$=\frac{8+14i+3(-1)}{4-9(-1)}$$

$$=\frac{8-3+14i}{4+9}$$

$$=\frac{5+14i}{13}$$

$$(x+yi)=\frac{5}{13}+\frac{14}{13}i$$

$\Rightarrow x=\dfrac{5}{13}$ and $y=\dfrac{14}{13}$

(ii) $(3-2i)(x+yi)=2(x-2yi)+2i-1$

$$3x+3yi-2xi-2yi^2=2x-4yi+2i-1$$

$$3x+(3y-2x)i-2y(-1)=2x-1+(2-4y)i$$

$$(3x+2y)+(3y-2x)i=(2x-1)+(2-4y)i$$

$\Rightarrow$ $3x+2y=2x-1$ ....(i) and

$3y-2x=2-4y$ ......(ii)

From (i) $3x-2x+2y=-1$

$x+2y=-1$ ....(iii)

From (ii) $-2x+3y+4y=2$

$-2x+7y=2$ ....(iv)

Multiplying (iii) by 2 and adding in (iv)

$$\cancel{2x}+4y=\cancel{-2}$$

$$\underline{-\cancel{2x}+7y=\cancel{2}}$$

$$11y=0$$

$$y=\frac{0}{11}$$

$$\boxed{y=0}$$

Putting value of y in (iii)

$$x+2y=-1$$

$$x+2(0)=-1$$

$$x+0=-1$$

$$\boxed{x=-1}$$

(iii) $(3+4i)^2-2(x-yi)=x+yi$

$$(3)^2+(4i)^2+2(3)(4i)-2x+2yi=x+yi$$

$$9+16i^2+24i-2x+2yi=x+yi$$

$$9+16(-1)+24i-2x+2yi=x+yi$$

$$9-16+24i-2x+2yi=x+yi$$

$$-7-2x+(24+2y)i=x+yi$$

$\Rightarrow$ $x=-7-2x$

$$x+2x=-7$$

$$3x=-7$$

$$\boxed{x=\frac{-7}{3}}$$

and $24+2y=y$

$$2y-y=-24$$

$$\boxed{y=-24}$$

OBJECTIVE

Q. Select the correct answer.

1. $\left(27x^{-1}\right)^{\frac{-2}{3}} =$ ______

(a) $\frac{\sqrt[3]{x^2}}{9}$ (b) $\frac{\sqrt{x^3}}{9}$

(c) $\frac{\sqrt[3]{x^2}}{8}$ (d) $\frac{\sqrt{x^3}}{8}$

2. Write $\sqrt[7]{x}$ in exponential form ....

(a) x (b) x^7

(c) $x^{\frac{1}{7}}$ (d) $x^{\frac{7}{2}}$

3. Write $4^{\frac{2}{3}}$ with radical sign......

(a) $\sqrt[3]{4^2}$ (b) $\sqrt{4^3}$

(c) $\sqrt[2]{4^3}$ (d) $\sqrt{4^6}$

4. In $\sqrt[3]{35}$ the radicand is

(a) 3 (b) $\frac{1}{3}$

(c) 35 (d) None of these

5. $\left(\frac{25}{16}\right)^{\frac{-1}{2}} =$ ______

(a) $\frac{5}{4}$ (b) $\frac{4}{5}$

(c) $\frac{-5}{4}$ (d) $\frac{-4}{5}$

6. The conjugate of 5 + 4i is ______

(a) $-5+4i$ (b) $-5-4i$

(c) $5-4i$ (d) $5+4i$

7. The value of i^9 is _____

(a) 1 (b) -1

(c) i (d) $-i$

8. Every real number is _____

(a) A positive integer

(b) A rational number

(c) A negative integer

(d) A complex number

9. Real part of 2ab $(i+i^2)$ is ____

(a) 2ab (b) -2ab

(c) 2abi (d) -2abi

10. Imaginary part of $-i\,(3i+2)$ is____

(a) -2 (b) 2

(c) 3 (d) -3

11. Which of the following sets have the closure property w.r.t. addition ____

(a) $\{0\}$ (b) $\{0, -1\}$

(c) $\{0, 1\}$ (d) $\left\{1, \sqrt{2}, \frac{1}{2}\right\}$

12. Name the property of real numbers used in $\left(\frac{-\sqrt{5}}{2}\right)\times 1 = \frac{-\sqrt{5}}{2}\times 1$

(a) Additive identity

(b) Additive Inverse

(c) Multiplicative identity

(d) Multiplicative Inverse

13. If z < 0 then $x < y \Rightarrow$

(a) x z < y z (b) x z > y z

(c) x z = y z (d) none of these

14. If a, b $\in$ R then only one of a = b or a < b or a > b holds is called...

(a) Trichotomy property

(b) Transitive property

(c) Additive property

(d) Multiplicative property

15. A non-terminating, non-recurring decimal represents:
(a) A natural number
(b) A rational number
(c) An irrational number
(d) A prime number

16. The union of the set of rational numbers and irrational numbers is known as set of ____
(a) Rational number
(b) Irrational
(c) Real number
(d) Whole number

17. For each prime number A, $\sqrt{A}$ is an ____
(a) Irrational (b) Rational
(c) Real (d) Whole

18. Square roots of all positive non-square integers are ____
(a) Irrational (b) Rational
(c) Real (d) Whole

19. π is an ____ number.
(a) Irrational (b) Rational
(c) Real (d) None

20. $\forall$ a, b, c $\in \mathbb{R}$ than $a < b$ and $b < c$ $\Rightarrow a < c$ is ____ property.
(a) Transitive
(b) Trichotomy property
(c) Additive property
(d) Multiplicative property

21. Name the property of real numbers used in $x > y$ or $x = y$ or $x < y$.
(a) Trichotomy
(b) Transitive
(c) Additive
(d) Multiplicative

22. Name the property of real numbers used in $\pi + (-\pi) = 0$.
(a) Additive inverse
(b) Multiplicative inverse
(c) Additive identity
(d) Multiplicative identity

23. $\sqrt{3}.\sqrt{3}$ is a ____ number.
(a) Rational (b) Irrational
(c) Real (d) None

24. $\sqrt[n]{ab} =$ ____
(a) $\sqrt[n]{a}\,\sqrt[n]{b}$ (b) $\sqrt{a}\,\sqrt{b}$
(c) $\sqrt[n]{a}\,\sqrt{b}$ (d) $\sqrt{a}\,\sqrt[n]{b}$

25. $\sqrt[5]{-8} =$ ____
(a) $(-8)^{1/5}$ (b) $(-8)^5$
(c) (-8) (d) $(8)^{\frac{1}{5}}$

26. The value of i^{10} is:
(a) -1 (b) 1
(c) $-i$ (d) i

27. The solution set of $x^2 + 1 = 0$ is:
(a) $\{i, i\}$ (b) $\{i, -i\}$
(c) $\{-i, -i\}$ (d) None

28. The conjugate of $2 + 3i$ is ____
(a) $2 - 3i$ (b) $-2 - 3i$
(c) $-2 + 3i$ (d) $2 + 3i$

29. Real part of $\left(-1+\sqrt{-2}\right)^2$ is:
(a) -1 (b) $-2\sqrt{2}$
(c) 1 (d) $2\sqrt{2}$

30. Imaginary part of $\left(-1+\sqrt{-2}\right)^2$ is
(a) -1 (b) $-2\sqrt{2}$
(c) 1 (d) $2\sqrt{2}$

31. Product of a complex number and its conjugate is always a non-negative__

(a) Real (b) Irrational
(c) Rational (d) None

ANSWER KEY

1.	a	**2.**	c	**3.**	a	**4.**	c	**5.**	b
6.	c	**7.**	c	**8.**	d	**9.**	b	**10.**	a
11.	a	**12.**	c	**13.**	b	**14.**	a	**15.**	c
16.	c	**17.**	a	**18.**	a	**19.**	a	**20.**	a
21.	a	**22.**	a	**23.**	c	**24.**	a	**25.**	a
26.	a	**27.**	b	**28.**	a	**29.**	a	**30.**	b
31.	a								

REVIEW EXERCISE

3. Simplify: (i) $\sqrt[4]{81y^{-12}x^{-8}}$

$$=\left(3^4y^{-12}x^{-8}\right)^{\frac{1}{4}}$$

$$=\left(3^4\right)^{\frac{1}{4}}\left(y^{-12}\right)^{\frac{1}{4}}\left(x^{-8}\right)^{\frac{1}{4}}$$

$$=3y^{-3}x^{-2}$$

$$=\frac{3}{x^2y^3}$$

(ii) $\sqrt{25x^{10n}y^{8m}}$

$$=\left(5^2x^{10n}y^{8m}\right)^{\frac{1}{2}}$$

$$=\left(5^2\right)^{\frac{1}{2}}\left(x^{10n}\right)^{\frac{1}{2}}\left(y^{8m}\right)^{\frac{1}{2}}$$

$$=5x^{5n}y^{4m}$$

(iii) $\left(\frac{x^3y^4z^5}{x^{-2}y^{-1}z^{-5}}\right)^{\frac{1}{5}}$

$$=\left(x^{3+2}y^{4+1}z^{5+5}\right)^{\frac{1}{5}}$$

$$=\left(x^5y^5z^{10}\right)^{\frac{1}{5}}$$

$$=\left(x^5\right)^{\frac{1}{5}}\left(y^5\right)^{\frac{1}{5}}\left(z^{10}\right)^{\frac{1}{5}}$$

$$=xyz^2$$

(iv) $\left(\frac{32x^{-6}y^{-4}z}{625x^4yz^{-4}}\right)^{\frac{2}{5}}$

$$=\left(\frac{2^5x^{-6}y^{-4}z}{5^4x^4yz^{-4}}\right)^{\frac{2}{5}}$$

$$=\left(\frac{2^5x^{-6-4}y^{-4-1}z^{1+4}}{5^4}\right)^{\frac{2}{5}}$$

$$=\left(\frac{2^5x^{-10}y^{-5}z^5}{5^4}\right)^{\frac{2}{5}}$$

$$=\frac{\left(2^5\right)^{\frac{2}{5}}\left(x^{-10}\right)^{\frac{2}{5}}\left(y^{-5}\right)^{\frac{2}{5}}\left(z^5\right)^{\frac{2}{5}}}{\left(5^4\right)^{\frac{2}{5}}}$$

$$=\frac{2^2x^{-4}y^{-2}z^2}{5^{\frac{8}{5}}}$$

$$=\frac{4z^2}{x^4y^25.5^{\frac{3}{5}}}$$

Q.4. Simplify: $\sqrt{\frac{(216)^{\frac{2}{3}}\times(25)^{\frac{1}{2}}}{(0.04)^{\frac{-3}{2}}}}$

$$=\left[\frac{\left(2^3\times3^3\right)^{\frac{2}{3}}\times\left(5^2\right)^{\frac{1}{2}}}{\left(\frac{4}{100}\right)^{\frac{-3}{2}}}\right]^{\frac{1}{2}}$$

$$=\left[\frac{\left(2^3\right)^{\frac{2}{3}}\times\left(3^3\right)^{\frac{2}{3}}\times5}{\left(\frac{100}{4}\right)^{\frac{3}{2}}}\right]^{\frac{1}{2}}$$

$$=\left[\frac{2^2\times3^2\times5}{(25)^{\frac{3}{2}}}\right]^{\frac{1}{2}}=\left[\frac{2^2\times3^2\times5}{\left(5^2\right)^{\frac{3}{2}}}\right]^{\frac{1}{2}}$$

REVIEW EXERCISE

3. Simplify: (i) $\sqrt[4]{81y^{-12}x^{-8}}$

$$=\left(3^4y^{-12}x^{-8}\right)^{\frac{1}{4}}$$

$$=\left(3^4\right)^{\frac{1}{4}}\left(y^{-12}\right)^{\frac{1}{4}}\left(x^{-8}\right)^{\frac{1}{4}}$$

$$=3y^{-3}x^{-2}$$

$$=\frac{3}{x^2y^3}$$

(ii) $\sqrt{25x^{10n}y^{8m}}$

$$=\left(5^2x^{10n}y^{8m}\right)^{\frac{1}{2}}$$

$$=\left(5^2\right)^{\frac{1}{2}}\left(x^{10n}\right)^{\frac{1}{2}}\left(y^{8m}\right)^{\frac{1}{2}}$$

$$=5x^{5n}y^{4m}$$

(iii) $\left(\dfrac{x^3y^4z^5}{x^{-2}y^{-1}z^{-5}}\right)^{\frac{1}{5}}$

$$=\left(x^{3+2}y^{4+1}z^{5+5}\right)^{\frac{1}{5}}$$

$$=\left(x^5y^5z^{10}\right)^{\frac{1}{5}}$$

$$=\left(x^5\right)^{\frac{1}{5}}\left(y^5\right)^{\frac{1}{5}}\left(z^{10}\right)^{\frac{1}{5}}$$

$$=xyz^2$$

(iv) $\left(\dfrac{32x^{-6}y^{-4}z}{625x^4yz^{-4}}\right)^{\frac{2}{5}}$

$$=\left(\frac{2^5x^{-6}y^{-4}z}{5^4x^4yz^{-4}}\right)^{\frac{2}{5}}$$

$$=\left(\frac{2^5x^{-6-4}y^{-4-1}z^{1+4}}{5^4}\right)^{\frac{2}{5}}$$

$$=\left(\frac{2^5x^{-10}y^{-5}z^5}{5^4}\right)^{\frac{2}{5}}$$

$$=\frac{\left(2^5\right)^{\frac{2}{5}}\left(x^{-10}\right)^{\frac{2}{5}}\left(y^{-5}\right)^{\frac{2}{5}}\left(z^5\right)^{\frac{2}{5}}}{\left(5^4\right)^{\frac{2}{5}}}$$

$$=\frac{2^2x^{-4}y^{-2}z^2}{5^{\frac{8}{5}}}$$

$$=\frac{4z^2}{x^4y^2 5.5^{\frac{3}{5}}}$$

Q.4. Simplify: $\sqrt{\dfrac{(216)^{\frac{2}{3}}\times(25)^{\frac{1}{2}}}{(0.04)^{\frac{-3}{2}}}}$

$$=\left[\frac{(2^3\times3^3)^{\frac{2}{3}}\times(5^2)^{\frac{1}{2}}}{\left(\frac{4}{100}\right)^{\frac{-3}{2}}}\right]^{\frac{1}{2}}$$

$$=\left[\frac{(2^3)^{\frac{2}{3}}\times(3^3)^{\frac{2}{3}}\times5}{\left(\frac{100}{4}\right)^{\frac{3}{2}}}\right]^{\frac{1}{2}}$$

$$=\left[\frac{2^2\times3^2\times5}{(25)^{\frac{3}{2}}}\right]^{\frac{1}{2}}=\left[\frac{2^2\times3^2\times5}{(5^2)^{\frac{3}{2}}}\right]^{\frac{1}{2}}$$

$$=\left[\frac{2^2\times3^2\times5}{5^3}\right]^{\frac{1}{2}}=\left[\frac{2^2\times3^2}{5^2}\right]^{\frac{1}{2}}$$

$$=\frac{(2^2)^{\frac{1}{2}}\times(3^2)^{\frac{1}{2}}}{(5^2)^{\frac{1}{2}}}=\frac{2\times3}{5}=\frac{6}{5}$$

Q.5 Simplify:

$$\left(\frac{a^p}{a^q}\right)^{p+q}.\left(\frac{a^q}{a^r}\right)^{q+r}\div5\left(a^p.a^r\right)^{p-r}$$

$$=\left(a^{p-q}\right)^{p+q}.\left(a^{q-r}\right)^{q+r}\div5\left(a^{p+r}\right)^{p-r}$$

$$=a^{p^2-q^2}.a^{q^2-r^2}\div5a^{p^2-r^2}$$

$$=\frac{a^{p^2-q^2}.a^{q^2-r^2}}{5a^{p^2-r^2}}$$

$$=\frac{a^{p^2-q^2+q^2-r^2-p^2+r^2}}{5}$$

$$=\frac{a^0}{5}=\frac{1}{5}$$

Q.6. Simplify: $\left(\frac{a^{2l}}{a^{l+m}}\right)\left(\frac{a^{2m}}{a^{m+n}}\right)\left(\frac{a^{2n}}{a^{n+l}}\right)$

$$=a^{2l-l-m}\times a^{2m-m-n}\times a^{2n-n-l}$$

$$=a^{l-m}.a^{m-n}.a^{n-l}$$

$$=a^{l-m+m-n+n-l}$$

$$=a^0=1$$

Q.7 Simplify: $\sqrt[3]{\frac{a^l}{a^m}}\times\sqrt[3]{\frac{a^m}{a^n}}\times\sqrt[3]{\frac{a^n}{a^l}}$

$$=\left(\frac{a^l}{a^m}\right)^{\frac{1}{3}}\times\left(\frac{a^m}{a^n}\right)^{\frac{1}{3}}\times\left(\frac{a^n}{a^l}\right)^{\frac{1}{3}}$$

$$=\frac{a^{\frac{l}{3}}}{a^{\frac{m}{3}}}\times\frac{a^{\frac{m}{3}}}{a^{\frac{n}{3}}}\times\frac{a^{\frac{n}{3}}}{a^{\frac{l}{3}}}$$

$$=a^{\frac{l}{3}-\frac{m}{3}}\times a^{\frac{m}{3}-\frac{n}{3}}\times a^{\frac{n}{3}-\frac{l}{3}}$$

$$=a^{\frac{l}{3}-\frac{m}{3}+\frac{m}{3}-\frac{n}{3}+\frac{n}{3}-\frac{l}{3}}$$

$$=a^0=1$$

Unit 03

LOGARITHMS

Scientific Notation

A number written in the form $a\times10^n$, where $1\le a<10$ and n is an integer, is called the scientific notation.

Example

Write each of the following ordinary numbers in scientific notation

(i) 30600 (ii) 0.000058

Solution

(i) $30600 = 3.06\times10^4$

(move decimal point four places to the left)

(ii) $0.000058 = 5.8\times10^{-5}$

(move decimal point five places to the right)

Example

Change each of the following numbers from scientific notation to ordinary notation.

(i) 6.35×10^6 (ii) 7.61×10^{-4}

Solution

(i) $6.35\times10^6 = 6350000$

(move the decimal point six places to the right)

(ii) $7.61\times10^{-4} = 0.000761$

(move the decimal point four places to the left)

Exercise 3.1

Q1. Express each of the following numbers in scientific notation.

i) 5700

Sol: $5700 = 5.7\times10^3$ (move decimal point three places to left)

ii) 49,800,000

Sol: $49{,}800{,}000 = 4.98\times10^7$ (move decimal point seven places to left)

iii) 96,000,000

Sol: $96{,}000{,}000 = 9.6\times10^7$ (move decimal point seven places to left)

iv) 416.9

Sol: $416.9 = 4.169\times10^2$ (move decimal point two places to left)

v) 83,000

Sol: $83{,}000 = 8.3\times10^4$ (move decimal point four places to left)

vi) 0.00643

Sol: $0.00643 = 6.43\times10^{-3}$ (move decimal point three places to right)

vii) 0.0074

Sol: $0.0074 = 7.4\times10^{-3}$ (move decimal point three places to right)

viii) 60,000,000

Sol: $60{,}000{,}000 = 6.0\times10^7$ (move decimal point seven places to left)

ix) 0.00000000395

Sol: $0.00000000395 = 3.95\times10^{-9}$ (move decimal point nine places to right)

x) $\dfrac{275,000}{0.0025}$

Sol: $\dfrac{275,000}{0.0025}$

$= \dfrac{2.75\times10^{5}}{2.5\times10^{-3}}$ $\dfrac{\text{(move decimal point five places to left)}}{\text{(move decimal point three places to right)}}$

Q2. Express the following numbers in ordinary notation.

i) 6×10^{-4}

Sol: $6\times10^{-4} = 0.0006$ (move decimal point four places to left)

ii) 5.06×10^{10}

Sol: $5.06\times10^{10} = 50,600,000,000$ (move decimal point ten places to right)

iii) 9.018×10^{-6}

Sol: $9.018\times10^{-6} = 0.000009018$ (move decimal point six places to left)

iv) 7.865×10^{8}

Sol: $7.865\times10^{8} = 786,500,000$ (move decimal point eight places to right)

Logarithm of a Real Number

If $a^x = y$ then x is called the logarithm of y to the base 'a' and is written as $\log_a y = x$, where $a > 0$, $a \neq 1$ and $y > 0$

i.e., the logarithm of a number y to the base 'a' is the index x of the power to which a must be raised to get that number y.

The relations $a^x = y$ and $log_a y = x$ are equivalent. When one relation is given, it can be converted into the other. Thus

$$\boxed{a^x = y \Leftrightarrow \log_a y = x}$$

Example

Find $\log_4 2$, i.e., find log of 2 to the base 4.

Solution

Let $\log_4 2 = x$

Then its exponenetial form is $4^x = 2$

i.e., $2^{2x} = 2^1 \Rightarrow 2x = 1$

$\therefore\quad x = \dfrac{1}{2} \Rightarrow \log_4 2 = \dfrac{1}{2}$

Deductions from Definition of Logarithm

1. Since $a^0 = 1$, $\log_a 1 = 0$
2. Since $a^1 = a$, $\log_a a = 1$

Common Logarithm or Brigg's Logarithm

If the base of logarithm is taken as 10 then logarithm is called Common Logarithm.

Characteristic

The integral part of the logarithm of any number is called the characteristic.

Characteristic of Logarithm of a number > 1

The characteristic of the logarithm of a number greater than 1 is always one less than the number of digits in the integral part of the number.

When a number b is written in the scientific notation, i.e., in the form $b = a \times 10^n$ where $1 \leq a < 10$, the power of 10 i.e., n will give the characteristic of $\log b$.

Examples

Number	Scientific Notation	Characteristic of the Logarithm
1.02	1.02×10^0	0
99.6	9.96×10^1	1
102	1.092×10^2	2
1662.4	1.6624×10^3	3

Characteristic of Logarithm of a Number < 1

The characteristic of the logarithm of a number less than 1, is always negative and one more than the number of zeroes immediately after the decimal point of the number.

Example

Write the characteristic of the log of following numbers by expressing them in scientific notation and noting the power of 10.

0.872, 0.02, 0.00345

Number	Scientific Notation	Characteristic of the Logarithm
0.872	8.72×10^{-1}	-1
0.02	2.0×10^{-2}	-2
0.00345	3.45×10^{-3}	-3

Mantissa

The fractional part of the logarithm of a number is called the mantissa. Mantissa is always positive

Example

Find the mantissa of the logarithm of 43.254

Solution

Rounding off 43.254 we consider only the four significant digits 4325.

(i) We first locate the row corresponding to 43 in the log tables and

(ii) Proceed horizontally till we reach the column corresponding to 2. The number at the intersection is 6355.

(iii) Again proceeding horizontally till the mean difference column corresponding to 5 intersects this row, we get the number 5 at the intersection.

(iv) Adding the two numbers 6355 and 5 we get .6360 as the mantissa of the logarithm of 43.25.

Example

Find the mantissa of the logarithm of 0.002347

Solution

Here also, we consider only the four significant digits 2347

We first locate the row corresponding to 23 in the logarithm tables and proceed as before.

Along the same row to its intersection with the column corresponding to 4 the resulting number is 3692. The number at the intersection of this row and the mean difference column corresponding to 7 is 13. Hence the sum of 3692 and 13 gives the mantissa of the logarithm of 0.0023476 as 0.3705

Example

Find (i) log 278.23

(ii) log 0.07058

Solution

(i) 278.23 can be rounded off as 278.2

The characteristic is 2 and the mantissa, using log tables, is .4443

$\therefore$ log 278.23 = 2.4443

(ii) The characteristic of log 0.07058 is -2 which is written as $\bar{2}$ by convention.

Using log tables the mantissa is .8487, so that

$$\text{Log } 0.07058 = \bar{2}.8487$$

Example

Find the numbers whose logarithms are

(i) 1.3247 (ii) $\bar{2}.1324$

Solution

(i) 1.3247

Reading along the row corresponding to .32 (as mantissa = 0.3247), we get 2109 at the intersection of this row with the column corresponding to 4. The number at the intersection of this row and the mean difference column corresponding to 7 is 3. Adding 2109 and 3 we get 2112.

Since the characteristic is 1, it is increased by 1 (because there should be two digits in the integral part) and therefore the decimal point is fixed after two digits from left in 2112.

Hence antilog of 1.3247 is 21.12.

(ii) $\bar{2}.1324$

Proceeding as in (i) the significant figures corresponding to the mantissa 0.1324 are 1356. Since the characteristic is $\bar{2}$, its numerical value 2 is decreased by 1. Hence there will be one zero after the decimal point.

Hence antilog of $\bar{2}.1324$ is 0.01356.

Exercise 3.2

Q1. Find the common logarithm of the following numbers.

i) 232.92

232.92 can be rounded off as 232.9

Characteristic = 2

Mantissa = .3672

Hence log 232.92 = 2.3672

ii) 29.326

29.326 can be rounded off as 29.33

Characteristic = 1

Mantissa = .4673

Hence log 29.326 = 1.4673

iii) 0.00032

Characteristic = $\bar{4}$

Mantissa = .5051

Hence log 0.0032 = $\bar{4}.5051$

iv) 0.3206

Characteristic = $\bar{1}$

Mantissa = .5060

Hence log 0.3206 = $\bar{1}.5060$

Q2. If log 31.09 = 1.4926, find the values of following:

i) log 3.109

Sol: log 3.109

Characteristic = 0

Mantissa = .4926

So log 3.109 = 0.4926

ii) log 310.9

Sol: log 310.9

Characteristic = 2

Mantissa = .4926

So log 310.9 = 2.4926

iii) log 0.003109

Sol: log 0.003109

Characteristic = $\bar{3}$

Mantissa = .4926

So log 0.003109 = $\bar{3}.4926$

iv) log 0.3109

Sol: log 0.3109

Characteristic = $\bar{1}$

Mantissa = .4926

So log 0.3109 = $\bar{1}.4926$

Q3. Find the numbers whose common logarithms are:

i) 3.5621

let the number be x

log x = 3.5621

Characteristic = 3

Mantissa = .5621

x = antilog 3.5621 = 3648

x = 3648

Hence 3648 is the required number

ii) $\bar{1}.7427$

Let the number be x

Log x = $\bar{1}.7427$

Characteristic = $\bar{1}$

Mantissa = .7427

x = antilog $\bar{1}.7427$ = 0.5530

x = 0.5530

Hence 0.5530 is the required number

Q4. What replacement for the unknown in each of following will make the statement true?

i) $\log_3 81 = L$

In exponential form

$$3^L = 81$$

$$3^L = 3^4$$

$\Rightarrow$ $\boxed{L = 4}$ Bases are equal so exponents are equal

ii) $\log_a 6 = 0.5$

In exponential form

$$a^{0.5} = 6$$

$$a^{\frac{1}{2}} = 6$$

Squaring both side

$$\left(a^{\frac{1}{2}}\right)^2 = (6)^2$$

$$\boxed{a = 36}$$

iii) $\log_5 n = 2$

In exponential form

$$5^2 = n$$

$\Rightarrow$ $\boxed{n = 25}$

iv) $10^P = 40$

In logarithmic form

$\mathrm{Log}_{10} 40 = P$

or log 40 = P

Characteristic = 1

Mantissa = .6021

So, P = 1.6021

Q5. Evaluate

i) $\log_2 \frac{1}{128}$

Let $x = \mathrm{Log}_2 \frac{1}{128}$

In exponential form

$$2^x = \frac{1}{128}$$

$$2^x = \frac{1}{2^7}$$

$$2^x = 2^{-7}$$

$\Rightarrow$ $\boxed{x = -7}$

ii) log 512 to the base $2\sqrt{2}$

Sol: $\log_{2\sqrt{2}} 512$

Let $x = \log_{2\sqrt{2}} 512$

In exponential form

$$\left(2\sqrt{2}\right)^x = 512$$

$$\left(2\times 2^{\frac{1}{2}}\right)^x = 2^9$$

$$\left(2^{1+\frac{1}{2}}\right)^x = 2^9$$

$$\left(2^{\frac{3}{2}}\right)^x = 2^9$$

$$2^{\frac{3}{2}x} = 2^9$$

$$\Rightarrow \quad \frac{3}{2}x = 9$$

$$x = {}^{3}\cancel{9}\times\frac{2}{\cancel{3}}$$

$$\boxed{x=6}$$

Q6. Evaluate the value of 'x' from the following statements.

i) $\log_2 x = 5$

In exponential form

$$2^5 = x$$

$$\Rightarrow \quad \boxed{x=32}$$

ii) $\log_{81} 9 = x$

In exponential form

$$81^x = 9$$

$$\left(9^2\right)^x = 9$$

$$9^{2x} = 9^1$$

$$\Rightarrow \quad 2x = 1$$

$$\text{or} \quad \boxed{x=\frac{1}{2}}$$

iii) $\log_{64} 8 = \frac{x}{2}$

In exponential form

$$(64)^{\frac{x}{2}} = 8$$

$$\left(8^2\right)^{\frac{x}{2}} = 8$$

$$8^{\cancel{2}\times\frac{x}{\cancel{2}}} = 8$$

$$8^x = 8^1$$

$$\Rightarrow \quad \boxed{x=1}$$

iv) $\log_x 64 = 2$

In exponential form

$$x^2 = 64$$

$$x^2 = 8^2$$

$$\Rightarrow \quad \boxed{x=8}$$

v) $\log_3 x = 4$

In exponential form

$$3^4 = x$$

$$\Rightarrow \quad \boxed{x=81}$$

Laws of Logarithm

In this section we shall prove the laws of logarithm and then apply them to find products, quotients, powers and roots of numbers.

(i) $\log_a(mn) = \log_a m + \log_a n$

(ii) $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

(iii) $\log_a m^n = n\log_a m$

(iv) $\log_a n = \log_b n \times \log_a b$

or $= \dfrac{\log_b n}{\log_b a}$

(i) $\log_a(mn) = \log_a m + \log_a n$:

Proof

Let $\log_a m = x$ *and* $\log_a n = y$

Writing in exponential form

$a^x = m$ and $a^y = n$

$\therefore \quad a^x \times a^y = mn$

i.e., $\quad a^{x+y} = mn$

or $\log_a(mn) = x + y = \log_a m + \log_a n$

Hence $\quad \log_a(mn) = \log_a m + \log_a n$

Note

(i) $\log_a(mn) \neq \log_a m \times \log_a n$

(ii) $\log_a m + \log_a n \neq \log_a(m+n)$

(iii) $\log_a(mnp..) = \log_a m + \log_a n + \log_a p + ..$

The rule given above is useful in finding the product of two or more numbers using logarithms.

Example

Evaluate 291.3×42.36

Solution

Let $x = 291.3 \times 42.36$

Then $\log x = \log(291.3 \times 42.36)$

$= \log 291.3 + \log 42.36$

$(\log_a mn = \log_a m + \log_a n)$

$= 2.4643 + 1.6269 = 4.0912$

$x = \text{antilog } 4.0912 = 12340$

Example

Evaluate 0.2913×0.004236.

Solution

Let $y = 0.2913 \times 0.004236$

Then $\log y = \log 0.2913 + \log 0.004236$

$\log y = \bar{1}.4643 + \bar{3}.6269$

$\log y = \bar{3}.0912$

$y = \text{anti log } \bar{3}.0912$

$y = 0.001234$

(ii) $\log_a\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

Proof

Let $\log_a m = x$ and $\log_a n = y$

So that $a^x = m$ and $a^y = n$

$\therefore \quad \dfrac{a^x}{a^y} = \dfrac{m}{n} \Rightarrow a^{x-y} = \dfrac{m}{n}$

i.e.,

$\log_a\left(\dfrac{m}{n}\right) = x - y = \log_a m - \log_a n$

Hence $\log_a\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

Note

(i) $\log_a\left(\dfrac{m}{n}\right) \neq \dfrac{\log_a m}{\log_a n}$

(ii) $\log_a m - \log_a n \neq \log_a(m-n)$

(iii) $\log_a\left(\dfrac{1}{n}\right) = \log_a 1 - \log_a n = -\log_a n .....$

$(\because \log_a 1 = 0)$

Example

Evaluate $\dfrac{291.3}{42.36}$

Solution

Let $x = \dfrac{291.3}{42.36}$ so that $\log x = \log \dfrac{291.3}{42.36}$

Then $\log x = \log 291.3 - \log 42.36, ...$

$(\log_a \dfrac{m}{n} = \log_a m - \log_a n)$

$\log x = 2.4643 - 1.6269 = 0.8374$

Thus $x = \text{antilog } 0.8374 = 6.877$

Example

Evaluate $\dfrac{0.0002913}{0.04236}$

Solution

Let $y = \dfrac{0.0002913}{0.04236}$ so that

$\log y = \log\left(\dfrac{0.002913}{0.04236}\right)$

then $\log y = \log 0.002913 - \log 0.04236$

$$\log y = \bar{3}.4643 - \bar{2}.6269$$
$$= \bar{3} + (0.4643 - 0.6269) - \bar{2}$$
$$= \bar{3} - 0.1626 - \bar{2}$$
$$= \bar{3} + (1 - 0.1626) - 1 - \bar{2},$$
(adding and subtracting 1)
$$= \bar{2}.8374$$
$[\therefore\ \bar{3} - 1 - \bar{2} = -3 - 1 - (-2) = -2 = \bar{2}]$

Therefore, $y = \text{antilog } \bar{2}.8374$

$y = 0.06877$

(iii) $\log_a(m^n) = n\log_a m$:

Proof

Let $\log_a m^n = x$, i.e., $a^x = m^n$

and $\log_a m = y$, i.e., $a^y = m$

Then $a^x = m^n = (a^y)^n$

i.e., $a^x = (a^y)^n = a^{yn} \Rightarrow x = ny$

i.e., $\log_a m^n = n\log_a m$

Example

Evaluate $\sqrt[4]{(0.0163)^3}$

Solution

Let $y = \sqrt[4]{(0.0163)^3} = (0.0163)^{3/4}$

$$\log y = \frac{3}{4}(\log 0.0163)$$
$$= \frac{3}{4} \times \bar{2}.2122$$
$$= \frac{\bar{6}.6366}{4}$$
$$= \frac{\bar{8} + 2.6366}{4}$$
$$= \bar{2} + 0.6592 = \bar{2}.6592$$

Hence $y = \text{antilog } \bar{2}.6592$

$= 0.04562$

(iv) Change of Base Formula:

$$\log_a n = \log_b n \times \log_a b \text{ or } \frac{\log_b n}{\log_b a}$$

Proof

Let $\log_b n = x$ so that $n = b^x$

Taking log to the base a, we have

$$\log_a n = \log_a b^x = x\log_a b = \log_b n \log_a b$$

Thus $\log_a n = \log_b n \log_a b$ ......(i)

Putting $n = a$ in the above result, we get

$$\log_b a \times \log_a b = \log_a a = 1$$

or $\log_a b = \dfrac{1}{\log_b a}$

Hence equation (i) gives

$$\log_a n = \frac{\log_b n}{\log_b a} \quad ......(ii)$$

Using the above rule, a natural logarithm can be converted to a common logarithm and vice versa.

$$\log_e n = \log_{10} n \times \log_e 10 \text{ or } \frac{\log_{10} n}{\log_{10} e}$$

$$\log_{10} n = \log_e n \times \log_{10} e \text{ or } \frac{\log_e n}{\log_e 10}$$

The values of $\log_e 10$ and $\log_{10} e$ are available from the tables:

$$\log_e 10 = \frac{1}{0.4343} = 2.3026 \quad and$$

$$\log_{10} e = \log 2.718 = 0.4343$$

Example

Calculate $\log_2 3 \times \log_3 8$

Solution

We know that

$$\log_a n = \frac{\log_b n}{\log_b a}$$

$$\therefore \quad \log_2 3 \times \log_3 8 = \frac{\log 3}{\log 2} \times \frac{\log 8}{\log 3}$$

$$= \frac{\log 8}{\log 2} = \frac{\log 2^3}{\log 2}$$

$$= \frac{3\log 2}{\log 2} = 3$$

Exercise 3.3

Q1. Write the following into sum or difference.

i) $\log(A \times B)$

Sol: $\log(A \times B) = \log A + \log B$

ii) $\log \frac{15.2}{30.5}$

Sol: $\log \frac{15.2}{30.5} = \log 15.2 - \log 30.5$

iii) $\log \frac{21 \times 5}{8}$

Sol: $\log \frac{21 \times 5}{8} = \log 21 + \log 5 - \log 8$

iv) $\log \sqrt[3]{\frac{7}{15}}$

Sol: $\log \sqrt[3]{\frac{7}{15}} = \log \left(\frac{7}{15}\right)^{\frac{1}{3}} = \frac{1}{3} \log \left(\frac{7}{15}\right)$

$= \frac{1}{3}(\log 7 - \log 15)$

v) $\log \frac{(22)^{\frac{1}{3}}}{5^3}$

Sol: $\log \frac{(22)^{\frac{1}{3}}}{5^3} = \log (22)^{\frac{1}{3}} - \log 5^3$

$= \frac{1}{3} \log 22 - 3 \log 5$

vi) $\log \frac{25 \times 47}{29}$

$= \log 25 + \log 47 - \log 29$

Q2. Express

$\log x - 2\log x + 3\log(x+1) - \log(x^2-1)$

as a single logarithm

Sol:

$\log x - 2\log x + 3\log(x+1) - \log(x^2-1)$

$= \log x - \log x^2 + \log(x+1)^3 - \log(x^2-1)$

$= \log x + \log(x+1)^3 - \log x^2 - \log(x^2-1)$

$= \log \frac{x(x+1)^3}{x^2(x^2-1)}$

$= \log \frac{(x+1)^3}{x(x-1)(x+1)}$

$= \log \frac{(x+1)^2}{x(x-1)}$

Q3. Write the following in the form of a single logarithm.

i) $\log 21 + \log 5$

Sol: $\log 21 + \log 5$

$= \log 21 \times 5$

ii) $\log 25 - 2\log 3$

$= \log 25 - \log 3^2$

$= \log \frac{25}{3^2} = \log \frac{25}{9}$

iii) $2\log x - 3\log y$

Sol: $2\log x - 3\log y$

$= \log x^2 - \log y^3$

$= \log \frac{x^2}{y^3}$

iv) $\log 5 + \log 6 - \log 2$

Sol: $\log 5 + \log 6 - \log 2$

$= \log \frac{5 \times 6}{2}$

Q4. Calculate the following:

i) $\log_3 2\times\log_2 81$

Sol: As we know that $\log_a n=\frac{\log_b n}{\log_b a}$

$\therefore \log_3 2\times\log_2 81=\frac{\log 2}{\log 3}\times\frac{\log 81}{\log 2}$

$=\frac{\log 81}{\log 3}$

$=\frac{\log 3^4}{\log 3}$

$=\frac{4\log 3}{\log 3}$

$=4$

ii) $\log_5 3\times\log_3 25$

Sol: As we know that

$\log_a n=\frac{\log_b n}{\log_b a}$

$\log_5 3\times\log_3 25=\frac{\cancel{\log 3}}{\log 5}\times\frac{\log 25}{\log 3}$

$=\frac{\log 25}{\log 5}$

$=\frac{\log 5^2}{\log 5}$

$=\frac{2\cancel{\log 5}}{\cancel{\log 5}}$

$=2$

Q5. If $\log 2=0.3010$, $\log 3=0.4771$, $\log 5=0.6990$**, then find the values of the following.**

i) $\log 32$

Sol: $\log 32$

$=\log 2^5$

$=5\log 2$

$=5(0.3010)$

$=1.5050$

ii) $\log 24$

$=\log 8\times 3$

$=\log 2^3\times 3$

$=\log 2^3+\log 3$

$=3\log 2+\log 3$

$=3(0.3010)+0.4771$

$=0.9030+0.4771$

$=1.3801$

iii) $\log\sqrt{3\frac{1}{3}}$

$=\log\sqrt{\frac{10}{3}}$

$=\log\left(\frac{2\times 5}{3}\right)^{\frac{1}{2}}$

$=\frac{1}{2}\log\left(\frac{2\times 5}{3}\right)=\frac{1}{2}(\log 2+\log 5-\log 3)$

$=\frac{1}{2}(0.3010+0.6990-0.4771)$

$=\frac{1}{2}(0.5229)$

$=0.2615$

iv) $\log\frac{8}{3}$

$=\log\frac{2^3}{3}$

$=\log 2^3-\log 3$

$=3\log 2-\log 3$

$=3(0.3010)-0.4771$

$=0.4259$

v) $\log 30$

$=\log 2\times 3\times 5$

$=\log 2+\log 3+\log 5$

$=0.3010+0.4771+0.6990$

$=1.4771$

Applications of logarithm

Example

Show that

$$7\log\frac{16}{15}+5\log\frac{25}{24}+\log\frac{81}{80}=\log 2.$$

Solution

$$\text{L.H.S} = 7\log\frac{16}{15}+5\log\frac{25}{24}+\log\frac{81}{80}$$

$$= 7[\log 16-\log 15]+5[\log 25-\log 24] +3[\log 81-\log 80]$$

$$= 7[\log 2^4-\log(3\times5)]+5[\log 5^2-\log (2^3\times3)]+3[\log 3^4-\log(2^4\times5)]$$

$$= 7[4\log 2-\log 3-\log 5]+5[2\log 5-3\log 2-\log 3] +3[4\log 3-4\log 2-\log 5]$$

$$= (28-15-12)\log 2+(-7-5+12)\log 3+(-7+10-3)\log 5$$

$$= \log 2+0+0=\log 2 = \text{R.H.S}$$

Example

Evaluate:

$$\sqrt[3]{\frac{0.07921\times(18.99)^2}{(5.79)^4\times0.9474}}$$

Let y =

$$\sqrt[3]{\frac{0.07921\times(18.99)^2}{(5.79)^4\times0.9474}} =$$

$$\left(\frac{0.07921\times(18.99)^2}{(5.79)^4\times0.9474}\right)^{1/3}$$

Log y =

$$\frac{1}{3}\log\left(\frac{0.07921\times(18.99)^2}{(5.79)^4\times0.9474}\right)$$

$$=\frac{1}{3}[\log\{0.07921\times(18.99)^2\}-\log\{(5.79)^4\times0.9474\}]$$

$$=\frac{1}{3}[\log 0.07921+2\log 18.99-4\log 5.79-\log 0.9474\}$$

$$=\frac{1}{3}[\bar{2}.8988+2(1.2786)-4(0.7627)-\bar{1}.9765]$$

$$=\frac{1}{3}[\bar{2}.8988+2.5572-3.0508-\bar{1}.9765]$$

$$=\frac{1}{3}[-2+0.8988+2.5572-3.0508+1-0.9765]$$

$$=\frac{1}{3}(\bar{2}.4287)$$

$$=\frac{1}{3}(\bar{3}+1.4287)$$

$$=\bar{1}+0.4762=\bar{1}.4762$$

$$y = \text{antilog } \bar{1}.4762=0.2993$$

Example

Given $A = A_o e^{-kd}$. If k = 2, what should be the value of d to make $A = \frac{A_o}{2}$?

Solution

Given that $A = A_o e^{-kd}$ $\Rightarrow$

$$\frac{A}{A_o} = e^{-kd}$$

Substituting k = 2 and $A = \frac{A_o}{2}$,

we get $\frac{1}{2} = e^{-2d}$

Taking common log on both sides,

$\log_{10} 1 - \log_{10}2 = -2d \log_{10}e$,

where $e = 2.718$

$$0 - 0.3010 = -2d\ (0.4343)$$

$$d = \frac{0.3010}{2\times0.4343}=0.3465$$

Exercise 3.4

Q1. Use log tables to find the values of

i) 0.8176×13.64

Sol: Let $x = 0.8176 \times 13.64$

Taking log of both sides

$\log x = \log 0.8176 \times 13.64$

$\log x = \log 0.8176 + \log 13.64$

$= \bar{1}.9125 + 1.1348$

$= -1 + 0.9125 + 1.1348$

$\log x = 1.0473$

Characteristics	=	1
Mantissa	=	.0473
x = antilog 1.0473	=	11.15

ii) $(789.5)^{\frac{1}{8}}$

Sol: Let $x = (789.5)^{\frac{1}{8}}$

Taking log of both sides

$\log x = \log(789.5)^{\frac{1}{8}}$

$= \frac{1}{8}\log(789.5)$

$= \frac{1}{8}(2.8974)$

$\log x = 0.3622$

Characteristics	=	0
Mantissa	=	.3622
x = antilog 0.3622	=	2.302

iii) $\dfrac{0.678 \times 9.01}{0.0234}$

Let $x = \dfrac{0.678 \times 9.01}{0.0234}$

Taking log of both sides

$\log x = \log \dfrac{0.678 \times 9.01}{0.0234}$

$= \log 0.678 + \log 9.01 - \log 0.0234$

$= \bar{1}.8312 + 0.9547 - (\bar{2}.3692)$

$= -1 + 0.8312 + 0.9547 - (-2 + 0.3692)$

$= -1 + 0.8312 + 0.9547 + 2 - 0.3692$

$\log x = 2.4167$

Characteristics	=	2
Mantissa	=	.4167
x = antilog 2.4167	=	261.0

iv) $\sqrt[5]{2.709} \times \sqrt[7]{1.239}$

Sol: Let $x = \sqrt[5]{2.709} \times \sqrt[7]{1.239}$

Taking log of both sides

$\log x = \log(2.709)^{\frac{1}{5}} \times (1.239)^{\frac{1}{7}}$

$= \log(2.709)^{\frac{1}{5}} + \log(1.239)^{\frac{1}{7}}$

$= \frac{1}{5}\log(2.709) + \frac{1}{7}\log(1.239)$

$= \frac{1}{5}(0.4328) + \frac{1}{7}(0.0931)$

$= 0.0866 + 0.0133$

$\log x = 0.0999$

Characteristics = 0

Mantissa = .0999

$x = \text{antilog } 0.0999$

$x = 1.259$

v) $\dfrac{(1.23)(0.6975)}{(0.0075)(1278)}$

Sol: Let $x = \dfrac{(1.23)(0.6975)}{(0.0075)(1278)}$

Taking log of both sides

$\log x = \log \dfrac{(1.23)(0.6975)}{(0.0075)(1278)}$

$= \log 1.23 + \log 0.6975 - \log 0.0075 - \log 1278$

$= 0.0899 + \bar{1}.8435 - \bar{3}.8751 - 3.1065$

$= 0.0899 - 1 + 0.8435 + 3 - 0.8751 - 3.1065$

$\log x = -1.0482$

$= -2 + 2 - 1.0482$

$= -2 + 0.9518$

$\log x = \bar{2}.9518$

Characteristics $= \bar{2}$

Mantissa $= .9518$

$x =$ antilog $\bar{2}.9518 = 0.0895$

vi) $\sqrt[3]{\dfrac{0.7214 \times 20.37}{60.8}}$

Let $x = \sqrt[3]{\dfrac{0.7214 \times 20.37}{60.8}}$

$$x = \left(\frac{0.7214 \times 20.37}{60.8}\right)^{\frac{1}{3}}$$

Taking log of both sides

$$\log x = \log\left(\frac{0.7214 \times 20.37}{60.8}\right)^{\frac{1}{3}}$$

$$= \frac{1}{3}\log\left(\frac{0.7214 \times 20.37}{60.8}\right)$$

$$= \frac{1}{3}(\log 0.7214 + \log 20.37 - \log 60.8)$$

$$= \frac{1}{3}(\bar{1}.8582 + 1.3090 - 1.7839)$$

$$= \frac{1}{3}(-1 + 0.8582 + 1.3090 - 1.7839)$$

$$= \frac{1}{3}(-0.6167)$$

$\log x = -0.2056$

$= -1 + 1 - 0.2056$

$= -1 + 0.7944$

$\log x = \bar{1}.7944$

Characteristics $= \bar{1}$

Mantissa $= .7944$

$x =$ antilog $\bar{1}.7944$

$= 0.6229$

vii) $\dfrac{83 \times \sqrt[3]{92}}{127 \times \sqrt[5]{246}}$

Sol: Let $x = \dfrac{83 \times \sqrt[3]{92}}{127 \times \sqrt[5]{246}}$

$$x = \frac{83 \times (92)^{\frac{1}{3}}}{127 \times (246)^{\frac{1}{5}}}$$

Taking log of both sides

$$\log x = \log \frac{83 \times (92)^{\frac{1}{3}}}{127 \times (246)^{\frac{1}{5}}}$$

$$= \log 83 + \log (92)^{\frac{1}{3}} - \log 127 - \log (246)^{\frac{1}{5}}$$

$$= \log 83 + \frac{1}{3}\log(92) - \log 127 - \frac{1}{5}\log(246)$$

$$= 1.9191 + \frac{1}{3}(1.9638) - 2.1038 - \frac{1}{5}(2.391)$$

$$= 1.9191 + 0.6546 - 2.1038 - 0.4782$$

$\log x = -0.0083$

$= -1 + 1 - 0.0083$

$= -1 + 0.9917$

$\log x = \bar{1}.9917$

Characteristics $= \bar{1}$

Mantissa $= .9917$

$x =$ antilog $\bar{1}.9917 = 0.9811$

viii) $\dfrac{(438)^3 \sqrt{0.056}}{(388)^4}$

Sol: Let $x = \dfrac{(438)^3 \sqrt{0.056}}{(388)^4}$

$$x = \frac{(438)^3 \times (0.056)^{\frac{1}{2}}}{(388)^4}$$

Taking log of both sides

$$\log x = \log \frac{(438)^3 \times (0.056)^{\frac{1}{2}}}{(388)^4}$$

$$= \log(438)^3 + \log(0.056)^{\frac{1}{2}} - \log(388)^4$$

$$= 3\log(438) + \frac{1}{2}\log(0.056) - 4\log(388)$$

$$= 3(2.6415) + \frac{1}{2}(\bar{2}.7482) - 4(2.5888)$$

$$= 3(2.6415) + \frac{1}{2}(-2 + 0.7482) - 4(2.5888)$$

$$= 7.9245 + \frac{1}{2}(-1.2518) - 10.3552$$

$$= 7.9245 - 0.6259 - 10.3552$$

$$\log x = -3.0566$$

$$= -4 + 4 - 3.0566$$

$$= -4 + 0.9434$$

$$\log x = \bar{4}.9434$$

Characteristic = $\bar{4}$

Mantissa = .9434

$x =$ antilog $\bar{4}.9434$ = 0.0008778

Q2. A gas is expanding according to the law $PV^n = C$. Find C when P=80, V=3.1 and $n = \frac{5}{4}$.

Sol: $PV^n = C$

Taking log of both sides:

$$\log PV^n = \log C$$

$$\log P + \log V^n = \log C$$

$$\log C = \log P + n\log V$$

Putting $P = 80$, $V = 3.1$ and $n = \frac{5}{4}$

$$\log C = \log 80 + \frac{5}{4}\log 3.1$$

$$= 1.9031 + \frac{5}{4}(0.4914)$$

$$= 1.9031 + 0.6143$$

$$\log C = 2.5174$$

Characteristic = 2

Mantissa = .5174

C = antilog 2.5174

C = 329.2 unit

Q3. The formula $p = 90(5)^{-\frac{q}{10}}$ applies to the demand of a product, where 'q' is the number of units and p is the price of one unit. How many units will be demanded if the price is Rs. 18.00?

Sol: $p = 90(5)^{-\frac{q}{10}}$

$q = ?$ and $p = Rs.\ 18.00$

As $p = 90(5)^{-\frac{q}{10}}$

$$18 = 90(5)^{\frac{-q}{10}}$$

Taking log of both sides

$$\log 18 = \log 90(5)^{-\frac{q}{10}}$$

$$\log 18 = \log 90 + \log(5)^{-\frac{q}{10}}$$

$$\log 18 - \log 90 = \frac{-q}{10}\log 5$$

$$10(\log 18 - \log 90) = -q\log 5$$

$$10(1.2553 - 1.9542) = -q(0.6990)$$

$$-6.989 = -q(0.6990)$$

$$\Rightarrow \quad q(0.6990) = 6.989$$

$$q = \frac{6.989}{0.6990}$$

$$q = 9.998$$

$q = 10$ approximately

So 10 units will be demanded

OR

$$p = 90\ (5)^{-q/10}$$

Taking log of both sides

$$\log p = \log 90\ (5)^{-q/10}$$

$$\log p = \log 90 + \log\ (5)^{-q/10}$$

$$\log p = \log 90\ -\frac{q}{10}\ \log 5$$

$$\frac{q}{10}\ \log 5 = \log 90 - \log p$$

$$\frac{q}{10}\ \log 5 = \log 90 - \log 18$$

$$\frac{q}{10}\ \log 5 = \log\ \frac{90}{18}$$

$$\frac{q}{10}\ \log 5 = \log\ 5$$

$$\frac{q}{10} = \frac{\log 5}{\log 5}$$

$$\frac{q}{10} = 1$$

$q = 10$ Units

Q4. **If** $A = \pi r^2$

$$\pi = \frac{22}{7},\ r = 15,\ A = ?$$

As $A = \pi r^2$

Taking log of both sides

$$\log A = \log \pi r^2$$

$$= \log \pi + \log r^2$$

$$= \log \pi + 2\log r$$

$$= \log \frac{22}{7} + 2\log 15$$

$$= \log 22 - \log 7 + 2\log 15$$

$$= 1.3424 - 0.8451 + 2(1.1761)$$

$$= 1.3424 - 0.8451 + 2.3522$$

$$\log A = 2.8495$$

Characteristics = 2

Mantissa = .8495

A = antilog 2.8495

A = 707.1

Q5. **If** $v = \frac{1}{3}\pi r^2 h$, **find** v **when** $\pi = \frac{22}{7}, r = 2.5$ **and** $h = 4.2$

Sol: $v = \frac{1}{3}\pi r^2 h$

$$\pi = \frac{22}{7},\ r = 2.5 \quad \text{and}\ \ h = 4.2,\ v = ?$$

As $v = \frac{1}{3}\pi r^2 h$

Taking log of both sides

$$\log v = \log \frac{1}{3}\pi r^2 h$$

$$= \log \frac{1}{3} + \log \pi + \log r^2 + \log h$$

$$= \log \frac{1}{3} + \log \frac{22}{7} + 2\log r + \log h$$

$$= \log 1 - \log 3 + \log 22 - \log 7 + 2\log 2.5 + \log 4.2$$

$$= 0 - 0.4771 + 1.3424 - 0.8451 + 2(0.3979) + 0.6232$$

$$\log v = 1.4392$$

Characteristics = 1

Mantissa = .4392

$v =$ antilog 1.4392

$v =$ 27.49

Review Exercise 3

Q3. Find the value of 'x' in the following.

i) $\log_3 x = 5$

Sol. $\log_3 x = 5$

In exponential form

$x = 3^5$

$\Rightarrow$ $x = 243$

ii) $\log_4 256 = x$

Sol. $\log_4 256 = x$

In exponential form

$4^x = 256$

$4^x = 4^4$

$\Rightarrow$ $x = 4$

iii) $\log_{625} 5 = \frac{1}{4}x$

Sol. $\log_{625} 5 = \frac{1}{4}x$

In exponential form

$(625)^{\frac{1}{4}x} = 5$

$(5^4)^{\frac{1}{4}x} = 5$

$5^{4\times\frac{1}{4}x} = 5$

$5^x = 5^1$

$\Rightarrow$ $x = 1$

iv) $\log_{64} x = -\frac{2}{3}$

Sol. $\log_{64} x = -\frac{2}{3}$

In exponential form

$x = 64^{\frac{-2}{3}}$

$x = (4^3)^{\frac{-2}{3}}$

$= 4^{\cancel{3}\left(-\frac{2}{\cancel{3}}\right)}$

$x = 4^{-2}$

$x = \frac{1}{4^2}$

$x = \frac{1}{16}$

Q4. Find the value of 'x' in the following.

i) $\log x = 2.4543$

Characteristic = 2

Mantissa = .4543

x = antilog 2.4543

= 284.6

ii) $\log x = 0.1821$

Characteristic = 0

Mantissa = .1821

x = antilog 0.1821

= 1.521

iii) $\log x = 0.0044$

Characteristic = 0

Mantissa = .0044

x = antilog 0.0044

x = 1.010

iv) $\log x = \bar{1}.6238$

Characteristic = $\bar{1}$

Mantissa = .6238

x = antilog $\bar{1}.6238$

x = 0.4205

Q5. If log2 = 0.3010, log3 = 0.4771 and log 5 = 0.6990, then find the values of the following.

i) log45

Sol. log45

$= \log 3^2 \times 5$
$= \log 3^2 + \log 5$
$= 2\log 3 + \log 5$
$= 2(0.4771) + 0.6990$
$= 0.9542 + 0.6990$
$= 1.6532$

ii) $\log \frac{16}{15}$

$= \log \frac{2^4}{3 \times 5}$
$= \log 2^4 - \log 3 - \log 5$
$= 4\log 2 - \log 3 - \log 5$
$= 4(0.3010) - 0.4771 - 0.6990$
$= 1.2040 - 0.4771 - 0.6990$
$= 0.0279$

iii) log 0.048

$= \log \frac{48}{1000}$
$= \log \frac{16 \times 3}{10^3}$
$= \log \frac{2^4 \times 3}{2^3 \times 5^3}$
$= \log \frac{2 \times 3}{5^3}$
$= \log 2 + \log 3 - \log 5^3$
$= \log 2 + \log 3 - 3\log 5$
$= 0.3010 + 0.4771 - 3(0.6990)$
$= -1.3189$
$= -2 + 2 - 1.3189$
$= -2 + 0.6811$
$= \bar{2}.6811$

Q6. Simplify the following:

i) $\sqrt[3]{25.47}$

Sol. Let $x = (25.47)^{\frac{1}{3}}$
Taking log of both sides
$\log x = \log (25.47)^{\frac{1}{3}}$
$= \frac{1}{3}\log(25.47)$
$= \frac{1}{3}(1.4060)$
$\log x = 0.4687$
Characteristic = 0
Mantissa = .4687
x = antilog 0.4687
x = 2.942

ii) $\sqrt[5]{342.2}$

Sol. Let $x = (342.2)^{\frac{1}{5}}$
Taking log of both sides
$\text{Log } x = \log (342.2)^{\frac{1}{5}}$
$= \frac{1}{5}\log(342.2)$
$= \frac{1}{5}(2.5343)$
$\log x = 0.5069$
Characteristic = 0
Mantissa = .5069
x = antilog 0.5069
x = 3.213

iii) $\frac{(8.97)^3 \times (3.95)^2}{\sqrt[3]{15.37}}$

Sol: Let $x = \frac{(8.97)^3 \times (3.95)^2}{(15.37)^{\frac{1}{3}}}$
Taking log of both sides
$\log x = \log \frac{(8.97)^3 \times (3.95)^2}{(15.37)^{\frac{1}{3}}}$

$= \log(8.97)^3 + \log(3.95)^2 - \log(15.37)^{\frac{1}{3}}$

$= 3\log(8.97) + 2\log(3.95) - \frac{1}{3}\log(15.37)$

$= 3(0.9528) + 2(0.5966) - \frac{1}{3}(1.1867)$

$= 2.8584 + 1.1932 - 0.3956$

$\log x = 3.6560$

Characteristic = 3

Mantissa = .6560

x = antilog 3.6560

x = 4529

Objective

1. If $a^x = n$, then _____

(a) $a = \log_x n$ (b) $x = \log_n a$

(c) $x = \log_a n$ (d) $a = \log_n x$

2. The relation of $y = \log_z x$ implies

(a) $x^y = z$ (b) $z^y = x$

(c) $x^z = y$ (d) $y^z = x$

3. The logarithm of unity to any base is ___

(a) 1 (b) 10

(c) e (d) 0

4. The logarithm of any number to itself as base is___

(a) 1 (b) 0

(c) −1 (d) 10

5. log e = ____ where e ≈ 2. 718

(a) 0 (b) 0.4343

(c) ∞ (d) 1

6. The value of $\log\left(\frac{p}{q}\right)$ is ___

(a) log p −log q

(b) $\frac{\log p}{\log q}$

(c) log p + log q

(d) log q − log p

7. $\log m^n$ can be written as

(a) $(\log m)^n$ (b) m log n

(c) n log m (d) log (mn)

8. $\log_b a \times \log_c b$ can be written as___

(a) $\log_c a$ (b) $\log_a c$

(c) $\log_a b$ (d) $\log_b c$

9. $\log_y x$ will be equal to____

(a) $\frac{\log_z x}{\log_y z}$ (b) $\frac{\log_x z}{\log_y z}$

(c) $\frac{\log_z x}{\log_z y}$ (d) $\frac{\log_z y}{\log_z x}$

10. For common logarithm, the base is____

(a) 2 (b) 10

(c) e (d) 1

11. For natural logarithm, the base is___

(a) 10 (b) e

(c) 2 (d) 1

12. The integral part of the common logarithm of a number is called the___

(a) Characteristic (b) Mantissa

(c) Logarithm (d) None

13. The decimal part of the common logarithm of a number is called the _____:

(a) Characteristic (b) Mantissa

(c) Logarithm (d) None

14. If $x = \log y$, then y is called the ___ of x.
(a) Antilogarithm (b) Logarithm
(c) Characteristic (d) None

15. If the characteristic of the logarithm of a number is $\bar{2}$, that number will have zero (s) immediately after the decimal point.
(a) One (b) Two
(c) Three (d) Four

16. If the characteristic of the logarithm of a number is 1, that number will have ____ digits in its integral part
(a) 2
(b) 3
(c) 4
(d) 5

17. The value of x in $\log_3 x = 5$ is_____
(a) 243 (b) 143
(c) 200 (d) 144

18. The value of x in $\log x = 2.4543$ is
(a) 284.6 (b) 1.521
(c) 1.1010 (d) 0.4058

19. The number corresponding to a given logarithm is known as ___
(a) Logarithm (b) Antilogarithm
(c) Characteristic (d) None

20. 30600 in scientific notation is ___
(a) 3.06×10^4 (b) 3.006×10^4
(c) 30.6×10^4 (d) 306×10^4

21. 6.35×10^6 in ordinary notation is___
(a) 6350000 (b) 635000
(c) 6350 (d) 63500

22. A number written in the form $a \times 10^n$, where $1 \le a < 10$ and n is an integer is called ____
(a) Scientific notation
(b) Ordinary notation
(c) Logarithm notation
(d) None

23. $\log p - \log q$ is same as
(a) $\log\left(\frac{q}{p}\right)$
(b) $\log(p-q)$
(c) $\frac{\log p}{\log q}$
(d) $\log\left(\frac{p}{q}\right)$

ANSWER KEY

1.	c	2.	b	3.	d	4.	a	5.	b
6.	a	7.	c	8.	a	9.	c	10.	b
11.	b	12.	a	13.	b	14.	a	15.	a
16.	a	17.	a	18.	a	19.	b	20.	a
21.	a	22.	a	23.	d				

Unit 04

ALGEBRAIC EXPRESSIONS AND ALGEBRAIC FORMULAS

Define the following terms

Algebraic Expressions

When operations of addition and subtraction are applied to algebraic terms we obtain an algebraic expression. For instance, $5x^2-3x+\frac{2}{\sqrt{x}}$ and $3xy+\frac{3}{x}(x\neq 0)$ are algebraic expressions.

Polynomials

A polynomial in the variable x is an algebraic expression of the form

$$P(x)=a_nx^n+a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+...+a_1x+a_o,a_n\neq 0,....(i)$$

Where n, the highest power of x, is a non-negative integer called the degree of the polynomial and each coefficient a_n is a real number. The coefficient a_n of the highest power of x is called the leading coefficient of the polynomial. $2x^4y^3+x^2y^2+8x$ is a polynomial in two variables x and y and has degree 7.

Rational Express Expression

The quotient $\frac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$, where $q(x)$ is a non-zero polynomial, is called a rational expression.

For example, $\frac{2x+1}{3x+8}$, $3x+8\neq 0$ is a rational expression.

In the rational expression $\frac{p(x)}{q(x)}$, $p(x)$ is called the numerator and $q(x)$ is known as the denominator of the rational expression $\frac{p(x)}{q(x)}$. The rational expression $\frac{p(x)}{q(x)}$ need not be a polynomial.

Example

Reduce the following algebraic fractions to their lowest forms.

(i) $\frac{lx+mx-ly-my}{3x^2-3y^2}$ (ii) $\frac{3x^2+18x+27}{5x^2-45}$

Solution

(i) $\frac{lx+mx-ly-my}{3x^2-3y^2}$

$$= \frac{x(l+m)-y(l+m)}{3(x^2-y^2)}$$

$$= \frac{(l+m)(x-y)}{3(x+y)(x-y)}$$

$$= \frac{l+m}{3(x+y)}$$

Which is in the lowest forms.

(ii) $\dfrac{3x^2+18x+27}{5x^2-45}=\dfrac{3(x^2+6x+9)}{5(x^2-9)}$

$\dfrac{3(x+3)(x+3)}{5(x+3)(x-3)}$

$\dfrac{3(x+3)}{5(x-3)}$

Which is in the lowest forms

Example

Simplify (i) $\dfrac{1}{x-y}-\dfrac{1}{x+y}+\dfrac{2x}{x^2-y^2}$

(ii) $\dfrac{2x^2}{x^4-16}-\dfrac{x}{x^2-4}+\dfrac{1}{x+2}$

Solution

(i) $\dfrac{1}{x-y}-\dfrac{1}{x+y}+\dfrac{2x}{x^2-y^2}$

$=\dfrac{1}{x-y}-\dfrac{1}{x+y}+\dfrac{2x}{(x+y)(x-y)}$

$=\dfrac{x+y-(x-y)+2x}{(x+y)(x-y)}$

(L.C.M of denominators)

$=\dfrac{\cancel{x}+y-\cancel{x}+y+2x}{(x+y)(x-y)}$

$=\dfrac{2x+2y}{(x+y)(x-y)}$

$=\dfrac{2(x+y)}{(x+y)(x-y)}=\dfrac{2}{x-y}$

(ii) $\dfrac{2x^2}{x^4-16}-\dfrac{x}{x^2-4}+\dfrac{1}{x+2}$

$=\dfrac{2x^2}{(x^2+4)(x^2-4)}-\dfrac{x}{x^2-4}+\dfrac{1}{x+2}$

$=\dfrac{2x^2}{(x^2+4)(x+2)(x-2)}-\dfrac{x}{(x+2)(x-2)}+\dfrac{1}{x+2}$

$=\dfrac{2x^2-x(x^2+4)+(x^2+4)(x-2)}{(x^2+4)(x+2)(x-2)}=\dfrac{\cancel{2x^2}-\cancel{x^3}-\cancel{4x}+\cancel{x^3}+\cancel{4x}-\cancel{2x^2}-8}{(x^2+4)(x+2)(x-2)}$

$$= \frac{-8}{(x^2+4)(x+2)(x-2)}$$

$$= \frac{-8}{(x^2+4)(x^2-4)} = \frac{-8}{x^4-16}$$

Example

Find the product $\frac{x+2}{2x-3y} \cdot \frac{4x^2-9y^2}{xy+2y}$

Solution

$$\frac{x+2}{2x-3y} \cdot \frac{4x^2-9y^2}{xy+2y} = \frac{(x+2)[(2x)^2-(3y)^2]}{(2x-3y)(x+2)y}$$

$$= \frac{\cancel{(x+2)}(2x+3y)\cancel{(2x-3y)}}{y\cancel{(x+2)}\cancel{(2x-3y)}}$$

$$= \frac{2x+3y}{y}$$

Example

Simplify $\frac{7xy}{x^2-4x+4} \div \frac{14y}{x^2-4}$

Solution

$$\frac{7xy}{x^2-4x+4} \div \frac{14y}{x^2-4}$$

$$= \frac{7xy}{x^2-4x+4} \times \frac{x^2-4}{14y}$$

$$= \frac{7xy}{(x-2)\cancel{(x-2)}} \times \frac{(x+2)\cancel{(x-2)}}{14y}$$

$$= \frac{x(x+2)}{2(x-2)}$$

Example

Evaluate $\frac{3x^2\sqrt{y}+6}{5(x+y)}$ if x = –4 and y=9

Solution

We have, by putting $x = -4$ and $y = 9$,

$$\frac{3x^2\sqrt{y}+6}{5(x+y)} = \frac{3(-4)^2\sqrt{9}+6}{5(-4+9)} = \frac{3(16)(3)+6}{5(5)} = \frac{150}{25} = 6$$

Exercise 4.1

1. Identify whether the following algebraic expression are polynomials (yes or no).

(i)	$3x^2+\frac{1}{x}-5$	No
(ii)	$3x^3-4x^2-x\sqrt{x}+3$	No
(iii)	$x^2-3x+\sqrt{2}$	Yes
(iv)	$\frac{3x}{2x-1}+8$	No

2. State whether each of the following expression is a rational expression or not.

(i)	$\frac{3\sqrt{x}}{3\sqrt{x}+5}$	No
(ii)	$\frac{x^3-2x^2+\sqrt{3}}{2+3x-x^2}$	Yes
(iii)	$\frac{x^2+6x+9}{x^2-9}$	Yes

(iv) $\frac{2\sqrt{x}+3}{2\sqrt{x}-3}$ No

3. Reduce the following rational expression to the lowest forms.

(i) $\frac{120x^2y^3z^5}{30x^3yz^2}$

$= 4x^{2-3}y^{3-1}z^{5-2}$

$= 4x^{-1}y^2z^3$

$= \frac{4y^2z^3}{x}$

(ii) $\frac{8a(x+1)}{2(x^2-1)} = \frac{4a\cancel{(x+1)}}{(x-1)\cancel{(x+1)}} = \frac{4a}{x-1}$

(iii) $\frac{(x+y)^2-4xy}{(x-y)^2} = \frac{x^2+y^2+2xy-4xy}{(x-y)(x-y)}$

$= \frac{x^2+y^2-2xy}{(x-y)(x-y)}$

$= \frac{(x-y)^2}{(x-y)(x-y)}$

$= \frac{\cancel{(x-y)^2}}{\cancel{(x-y)^2}} = 1$

(iv) $\frac{(x^3-y^3)(x^2-2xy+y^2)}{(x-y)(x^2+xy+y^2)}$

$= \frac{\cancel{(x^3-y^3)}(x-y)^2}{\cancel{x^3-y^3}} = (x-y)^2$

(v) $\frac{(x+2)(x^2-1)}{(x+1)(x^2-4)}$

$= \frac{\cancel{(x+2)}(x-1)\cancel{(x+1)}}{\cancel{(x+1)}(x-2)\cancel{(x+2)}} = \frac{x-1}{x-2}$

(vi) $\frac{x^2-4x+4}{2x^2-8} = \frac{(x-2)^2}{2(x^2-4)}$

$= \frac{(x-2)^2}{2(x-2)(x+2)}$

$= \frac{\cancel{(x-2)}(x-2)}{2\cancel{(x-2)}(x+2)}$

$= \frac{x-2}{2(x+2)}$

(vii) $\frac{64x^5-64x}{(8x^2+8)(2x+2)}$

$= \frac{64x(x^4-1)}{8(x^2+1).2(x+1)}$

$= \frac{64x(x^4-1)}{16(x^2+1)(x+1)}$

$= \frac{4x(x^2+1)(x^2-1)}{(x^2+1)(x+1)}$

$= \frac{4x\cancel{(x^2+1)}(x-1)\cancel{(x+1)}}{\cancel{(x^2+1)}\cancel{(x+1)}}$

$= 4x(x-1)$

$\frac{9x^2-(x^2-4)^2}{4+3x-x^2} = \frac{(3x)^2-(x^2-4)^2}{4+3x-x^2}$

$= \frac{(3x+x^2-4)\cancel{(3x-x^2+4)}}{\cancel{(4+3x-x^2)}}$

$= 3x+x^2-4$

$= x^2+3x-4$

4. Evaluate (a) $\frac{x^3y-2z}{xz}$ for (i) x = 3, y = –1, z = –2.

(a) $\frac{(3)^3(-1)-2(-2)}{3(-2)} = \frac{-27+4}{-6}$

$= \frac{-23}{-6} = \frac{23}{6} = 3\frac{5}{6}$

(b) $\dfrac{x^2y^3-5z^4}{xyz}$ for $x = 4, y = -2, z = -1$

$$=\frac{(4)^2(-2)^3-5(-1)^4}{(4)(-2)(-1)}=\frac{-16(8)-5}{8}$$

$$=\frac{-128-5}{8}=\frac{-133}{8}=-16\frac{5}{8}$$

5. Perform the indicated operation and simplify

(i) $$\frac{15}{2x-3y}-\frac{4}{3y-2x}$$

$$=\frac{15(3y-2x)-4(2x-3y)}{(2x-3y)(3y-2x)}$$

$$=\frac{45y-30x-8x+12y}{(2x-3y)(3y-2x)}$$

$$=\frac{57y-38x}{(2x-3y)(3y-2x)}$$

$$=\frac{19\cancel{(3y-2x)}}{(2x-3y)\cancel{(3y-2x)}}=\frac{19}{2x-3y}$$

(ii) $$\frac{1+2x}{1-2x}-\frac{1-2x}{1+2x}$$

$$=\frac{(1+2x)^2-(1-2x)^2}{(1-2x)(1+2x)}$$

$$=\frac{(1+4x^2+4x)-(1+4x^2-4x)}{(1-2x)(1+2x)}$$

$$=\frac{\cancel{1}+\cancel{4x^2}+4x\cancel{-1}\cancel{-4x^2}+4x}{(1-2x)(1+2x)}$$

$$=\frac{8x}{(1-2x)(1+2x)}=\frac{8x}{1-4x^2}$$

(iii) $$\frac{x^2-25}{x^2-36}-\frac{x+5}{x+6}$$

$$=\frac{(x-5)(x+5)}{(x-6)(x+6)}-\frac{x+5}{x+6}$$

$$=\frac{(x-5)(x+5)-(x+5)(x-6)}{(x+6)(x-6)}$$

$$=\frac{(x+5)\left[(x-5)-(x-6)\right]}{(x+6)(x-6)}$$

$$=\frac{(x+5)(\cancel{x}-5\cancel{-x}+6)}{(x+6)(x-6)}$$

$$=\frac{(x+5)(1)}{(x+6)(x-6)}=\frac{x+5}{x^2-36}$$

(iv) $$\frac{x}{x-y}-\frac{y}{x+y}-\frac{2xy}{x^2-y^2}$$

$$=\frac{x(x+y)-y(x-y)}{(x-y)(x+y)}-\frac{2xy}{x^2-y^2}$$

$$=\frac{x^2+\cancel{xy}-\cancel{xy}+y^2}{x^2-y^2}-\frac{2xy}{x^2-y^2}$$

$$=\frac{x^2+y^2}{x^2-y^2}-\frac{2xy}{x^2-y^2}$$

$$=\frac{x^2+y^2-2xy}{(x^2-y^2)}$$

$$=\frac{(x-y)^{\cancel{2}}}{\cancel{(x-y)}(x+y)}=\frac{x-y}{x+y}$$

(v) $$\frac{x-2}{x^2+6x+9}-\frac{x+2}{2x^2-18}$$

$$=\frac{x-2}{x^2+3x+3x+9}-\frac{x+2}{2(x^2-9)}$$

$$=\frac{x-2}{x(x+3)+3(x+3)}-\frac{x+2}{2(x-3)(x+3)}$$

$$=\frac{x-2}{(x+3)(x+3)}-\frac{x+2}{2(x-3)(x+3)}$$

$$=\frac{2(x-3)(x-2)-(x+3)(x+2)}{2(x-3)(x+3)(x+3)}$$

$$=\frac{2(x^2-2x-3x+6)-(x^2+2x+3x+6)}{2(x-3)(x+3)^2}$$

$$=\frac{2(x^2-5x+6)-(x^2+5x+6)}{2(x-3)(x+3)^2}$$

$$=\frac{2x^2-10x+12-x^2-5x-6}{2(x-3)(x+3)^2}$$

$$=\frac{x^2-15x+6}{2(x-3)(x+3)^2}$$

(vi) $\frac{1}{x-1}-\frac{1}{x+1}-\frac{2}{x^2+1}-\frac{4}{x^4-1}$

$$=\frac{x+1-(x-1)}{(x-1)(x+1)}-\frac{2}{x^2+1}-\frac{4}{x^4-1}$$

$$=\frac{\cancel{x+1}-\cancel{x+1}}{x^2-1}-\frac{2}{x^2+1}-\frac{4}{x^4-1}$$

$$=\frac{2}{x^2-1}-\frac{2}{x^2+1}-\frac{4}{x^4-1}$$

$$=\frac{2(x^2+1)-2(x^2-1)}{(x^2-1)(x^2+1)}-\frac{4}{x^4-1}$$

$$=\frac{\cancel{2x^2}+2-\cancel{2x^2}+2}{x^4-1}-\frac{4}{x^4-1}$$

$$=\frac{4}{x^4-1}-\frac{4}{x^4-1}$$

$$=\frac{4-4}{x^4-1}$$

$$=\frac{0}{x^4-1}$$

$$=0$$

6. Perform the indicated operation and simplify:

(i) $(x^2-49)\,\frac{5x+2}{x+7}$

$$=(x-7)\cancel{(x+7)}\,\frac{5x+2}{\cancel{x+7}}$$

$$=(x-7)(5x+2)$$

(ii) $\frac{4x-12}{x^2-9}\div\frac{18-2x^2}{x^2+6x+9}$

$$=\frac{4(x-3)}{(x-3)(x+3)}\div\frac{2(9-x^2)}{x^2+3x+3x+9}$$

$$=\frac{4(x-3)}{(x-3)(x+3)}\div\frac{2(3-x)(3+x)}{x(x+3)+3(x+3)}$$

$$=\frac{4(x-3)}{(x-3)(x+3)}\div\frac{2(3-x)(3+x)}{(x+3)(x+3)}$$

$$=\frac{4(x-3)}{(x+3)(x-3)}\times\frac{(x+3)(x+3)}{2(3+x)(3-x)}$$

$$=\frac{2}{3-x}$$

(iii) $\frac{x^6-y^6}{x^2-y^2}\div(x^4+x^2y^2+y^4)$

$$=\frac{(x^3)^2-(y^3)^2}{x^2-y^2}\div(x^4+x^2y^2+y^4)$$

$$=\frac{(x^3-y^3)(x^3+y^3)}{x^2-y^2}\div(x^4+x^2y^2+y^4)$$

$$=\frac{(x-y)(x^2+xy+y^2)(x+y)(x^2-xy+y^2)}{x^2-y^2}\times\frac{1}{x^4+x^2y^2+y^4}$$

$$=\frac{\cancel{(x^2-y^2)}(x^2+xy+y^2)(x^2-xy+y^2)}{\cancel{x^2-y^2}}\times\frac{1}{x^4+x^2y^2+y^4}$$

$$=\frac{x^4+x^2y^2+y^4}{x^4+x^2y^2+y^4}=1$$

(iv) $\frac{x^2-1}{x^2+2x+1}\cdot\frac{x+5}{1-x}$

$$=\frac{-\cancel{(x-1)}(x+1)}{x^2+x+x+1}\cdot\frac{x+5}{(x-1)}$$

$$=\frac{-(x+1)(x+5)}{x(x+1)+1(x+1)}$$

$$=\frac{-\cancel{(x+1)}(x+5)}{(x+1)\cancel{(x+1)}}=-\frac{x+5}{x+1}$$

(v) $$\frac{x^2+xy}{y(x+y)}\cdot\frac{x^2+xy}{y(x+y)}\div\frac{x^2-x}{xy-2y}$$

$$=\frac{x\cancel{(x+y)}}{y\cancel{(x+y)}}\cdot\frac{\cancel{x}\cancel{(x+y)}}{\cancel{y}\cancel{(x+y)}}\times\frac{\cancel{y}(x-2)}{\cancel{x}(x-1)}$$

$$=\frac{x(x-2)}{y(x-1)}$$

Example

If $a+b=7$ and $a-b=3$, then find the value of (a) a^2+b^2 (b) ab

Solution

We are given that $a+b=7$ and $a-b=3$

(a) To find the value of (a^2+b^2), we use the formula

$$(a+b)^2+(a-b)^2=2(a^2+b^2)$$

Substituting the values $a+b=7$ and $a-b=3$, we get

$$(7)^2+(3)^2 = 2(a^2+b^2)$$

$$\Rightarrow 49+9 = 2(a^2+b^2)$$

$$\Rightarrow 58 = 2(a^2+b^2),$$

$$\Rightarrow 29 = a^2+b^2,$$

(b) To find the value of ab, we make use of the formula

$$(a+b)^2-(a-b)^2 = 4ab$$

$$\Rightarrow (7)^2-(3)^2 = 4ab,$$

$$\Rightarrow 49-9 = 4ab$$

$$\Rightarrow 40 = 4ab,$$

$$\Rightarrow 10 = ab,$$

Hence $a^2+b^2=29$ and $ab=10$.

Example

If $a^2+b^2+c^2=43$ and $ab+bc+ca=3$, then find the value of $a+b+c$.

Solution

We know that

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$$

$$(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$$

$$\Rightarrow (a+b+c)^2=43+2\times3$$

(Putting $a^2+b^2+c^2=43$ and $ab+bc+ca=3$)

$$\Rightarrow (a+b+c)^2=49$$

$$\Rightarrow a+b+c=\pm\sqrt{49}$$

Hence $a+b+c=\pm7$

Example

If $a+b+c=6$ and $a^2+b^2+c^2=24$ then find the value of $ab+bc+ca$.

Solution

We have

$$(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$$

$$(6)^2=24+2(ab+bc+ca)$$

$$\Rightarrow 36=24+2(ab+bc+ca)$$

$$\Rightarrow 12=2(ab+bc+ca)$$

Hence $ab+bc+ca=6$

Example

If $a+b+c=7$ and $ab+bc+ca=9$, then find the value of $a^2+b^2+c^2$.

Solution

We know that

$(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ca$

$\Rightarrow (a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$

$\Rightarrow\ (7)^2=a^2+b^2+c^2+2(9)$

$\Rightarrow\ 49=a^2+b^2+c^2+18$

$\Rightarrow\ 31=a^2+b^2+c^2$

Hence $a^2+b^2+c^2=31$

Example

If $2x-3y=10$ and $xy=2$, then find the value of $8x^3-27y^3$.

Solution

We are given that $2x-3y=10$

$\Rightarrow (2x-3y)^3 = (10)^3$

$\Rightarrow 8x^3-27y^3-3\times2x\times3y(2x-3y)=1000$

$\Rightarrow 8x^3-27y^3-18xy(2x-3y)=1000$

$\Rightarrow 8x^3-27y^3-18\times 2\times10 = 1000$

$\Rightarrow 8x^3-27y^3-360 = 1000$

Hence

$8x^3-27y^3 = 1360$

Example

If $x+\frac{1}{x}=8$, then find the value of $x^3+\frac{1}{x^3}$

Solution

We have been given $x+\frac{1}{x}=8$

$$\Rightarrow \left(x+\frac{1}{x}\right)^3 = (8)^3$$

$$\Rightarrow x^3+\frac{1}{x^3}+3\times x\times\frac{1}{x}\left(x+\frac{1}{x}\right)=512$$

$$\Rightarrow x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right) = 512$$

$$\Rightarrow x^3+\frac{1}{x^3}+3\times8 = 512$$

$$\Rightarrow x^3+\frac{1}{x^3}+24 = 512$$

$$\Rightarrow x^3+\frac{1}{x^3} = 512-24$$

$$\Rightarrow x^3+\frac{1}{x^3} = 488$$

Example

If $x-\frac{1}{x}=4$, then find $x^3-\frac{1}{x^3}$

Solution

We have $x-\frac{1}{x}=4$

$$\Rightarrow \left(x-\frac{1}{x}\right)^3 = (4)^3$$

$$\Rightarrow x^3-\frac{1}{x^3}-3x\times\frac{1}{x}\left(x-\frac{1}{x}\right) = 64$$

$$\Rightarrow x^3-\frac{1}{x^3}-3(4) = 64$$

$$\Rightarrow x^3-\frac{1}{x^3}-12 = 64$$

$$\Rightarrow x^3-\frac{1}{x^3} = 64+12$$

$$\Rightarrow x^3-\frac{1}{x^3} = 76$$

Example

Factorize $64x^3+343y^3$

Solution

We have

$64x^3+343y^3=(4x)^3+(7y)^3$

$= (4x+7y)[(4x)^2-(4x)(7y)+(7y)^2]$

$= (4x+7y)(16x^2-28xy+49y^2)$

Example

Factorize $125x^3-1331y^3$

Solution

We have

$125x^3-1331y^3 = (5x)^3-(11y)^3$

$= (5x-11y)[(5x)^2+(5x)(11y)+(11y)^2]$

$= (5x-11y)(25x^2+55xy+121y^2)$

Example

Factorize

$\left(\frac{2}{3}x+\frac{3}{2x}\right)\left(\frac{4}{9}x^2-1+\frac{9}{4x^2}\right)$

Solution

$\left(\frac{2}{3}x+\frac{3}{2x}\right)\left(\frac{4}{9}x^2-1+\frac{9}{4x^2}\right)$

$=\left(\frac{2}{3}x+\frac{3}{2x}\right)\left[\left(\frac{2}{3}x\right)^2-\left(\frac{2}{3}x\right)\left(\frac{3}{2x}\right)+\left(\frac{3}{2x}\right)^2\right]$

$=\left(\frac{2}{3}x\right)^3+\left(\frac{3}{2x}\right)^3$

$=\frac{8}{27}x^3+\frac{27}{8x^3}$

Example

Find the product $\left(\frac{4}{5}x-\frac{5}{4x}\right)\left(\frac{16}{25}x^2+\frac{25}{16x^2}+1\right)$

Solution

$\left(\frac{4}{5}x-\frac{5}{4x}\right)\left(\frac{16}{25}x^2+\frac{25}{16x^2}+1\right)$

$=\left(\frac{4}{5}x-\frac{5}{4x}\right)\left(\frac{16x^2}{25}+1+\frac{25}{16x^2}\right)$

(rearranging)

$=\left(\frac{4}{5}x-\frac{5}{4x}\right)\left[\left(\frac{4}{5}x\right)^2+\left(\frac{4}{5}x\right)\left(\frac{5}{4x}\right)+\left(\frac{5}{4x}\right)^2\right]$

$=\left(\frac{4}{5}x\right)^3-\left(\frac{5}{4x}\right)^3=\frac{64}{125}x^3-\frac{125}{64x^3}$

Example

Find the continued product of $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$

Solution

$(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$

$=(x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2)$

$=(x^3+y^3)(x^3-y^3)=(x^3)^2-(y^3)^2=x^6-y^6$

Exercise 4.2

1.(i) If a + b = 10 and a – b = 6 then find value of $a^2 + b^2$.

Solution:

$2(a^2+b^2) = (a+b)^2 + (a-b)^2$

$2(a^2+b^2) = (10)^2 + (6)^2$

$2(a^2+b^2) = 100 + 36$

$a^2 + b^2 = \frac{136}{2} = 68$

(ii) If a + b = 5, a – b = $\sqrt{17}$ then find value of ab.

Solution:

$4ab = (a+b)^2 - (a-b)^2$

$4ab = (5)^2 - (\sqrt{17})^2$

$4ab = 25 - 17$

$4ab = 8$

$$ab=\frac{8}{4}=2$$

2. If $a^2 + b^2 + c^2 = 45$ and $a + b + c = -1$ find value of $ab + bc + ca$.

Solution:

$a+b+c = -1$

Squaring

$(a+b+c)^2 = (-1)^2$

$a^2+b^2+c^2+2ab+abc+2ca = 1$

$a^2+b^2+c^2+2(ab+bc+ca) = 1$

$45 + 2(ab + bc + ca) = 1$

$2(ab + bc + ca) = 1 - 45$

$2(ab + bc + (a) = -44$

$ab + bc + ca = \frac{-44}{2} = -22$

3. If $m+n+p = 10$, $mn + np + pm = 27$ find value of $m^2+n^2+p^2$.

Solution:

$m+n+p = 10$

Squaring both sides

$(m+n+p)^2 = (10)^2$

$m^2+n^2+p^2+2mn+2np+2mp=100$

$m^2+n^2+p^2+2(mn+np+mp)=100$

$m^2+n^2+p^2+2(27) = 100$

$m^2+n^2+p^2+54 = 100$

$m^2+n^2+p^2 = 100 - 54$

$m^2+n^2+p^2 = 46$

4. If $x^2 + y^2 + z^2 = 78$ and $y+yz+zx=59$ find $x + y + z$.

Solution:

$(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx$

$= x^2+y^2+z^2+2(xy+yz+zx)$

$= 78+2(59)$

$= 78 + 118$

$= 196$

$\sqrt{(x+y+z)^{\cancel{2}}} = \sqrt{196} = \sqrt{(\pm 14)^2}$

$x+y+z = \pm 14$

5. If $x + y + z = 12$ and $x^2 + y^2 + z^2 = 64$ find value of $xy+yz+zx$.

Solution:

$x + y + z = 12$

Squaring both sides

$(x + y + z)^2 = (12)^2$

$x^2+y^2+z^2 + 2xy+2yz+2zx = 144$

$x^2 + y^2+z^2+2(xy+yz+zx) = 144$

$64 + 2(xy+yz+zx) = 144$

$2(xy + yz+zx) = 144 - 64$

$2(xy + yz+zx) = 80$

$xy + yz + zx = \frac{80}{2} = 40.$

6. If $x + y = 7$ and $xy = 12$ then find value of $x^3 + y^3$.

Solution:

$x + y = 7$

$(x + y)^3 = (7)^3$

$x^3 + y^3 + 3xy(x+y) = 343$

$x^3+y^3+3(12)(7) = 343$

$x^3 + y^3 + 252 = 343$

$x^3 + y^3 = 343 - 252$

$x^3+y^3 = 91$

7. If $3x + 4y = 11$ and $xy = 12$ then find value of $27x^3 + 64y^3$.

Solution:

$3x + 4y = 11$

$(3x + 4y)^3 = (11)^3$

$(3x)^3+(4y)^3+3(3x)(4x)(3x+4y)= 1331$

$27x^3+64y^3+36xy(3x+4y) = 1331$

$27x^3+64y^3+36(12)(11) = 1331$

$27x^3+64y^3+4752 = 1331$

$27x^3+64y^3 = 1331 - 4752 = -3421$

8. If $x - y = 4$ and $xy = 21$ then find value of $x^3 - y^3$.

Solution:

$x - y = 4$

$(x-y)^3 = (4)^3$

$x^3-y^3-3xy(x-y) = 64$

$x^3-y^3-3(21)(4) = 64$

$x^3-y^3-252 = 64$

$x^3 - y^3 = 64 + 252$

$x^3 - y^3 = 316$

9. If $5x - 6y = 13$ and $xy = 6$ then find value of $125x^3 - 216y^3$.

Solution:

$5x - 6y = 13$

$\Rightarrow (5x-6y)^3 = (13)^3$

$\Rightarrow (5x)^3 - (6y)^3 - 3(5x)(6y)(5x-6y) = 2197$

$125x^3 - 216y^3 - 90xy(5x-6y) = 2197$

$125x^3 - 216y^3 - 90(6)(13) = 2197$

$125x^3 - 216y^3 - 7020 = 2197$

$125x^3 - 216y^3 = 2197 + 7020$

$125x^3 - 216y^3 = 9217$

10. If $x + \frac{1}{x} = 3$ then find $x^3 + \frac{1}{x^3}$.

$x + \frac{1}{x} = 3$ Cubing both sides

$$\left(x + \frac{1}{x}\right)^3 = (3)^3$$

$$x^3 + \frac{1}{x^3} + 3(x)\left(\frac{1}{x}\right)\left(x + \frac{1}{x}\right) = 27$$

$$x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = 27$$

$$x^3 + \frac{1}{x^3} + 3(3) = 27$$

$$x^3 + \frac{1}{x^3} = 27 - 9$$

$$x^3 + \frac{1}{x^3} = 18$$

11. If $x - \frac{1}{x} = 7$, then find value of $x^3 - \frac{1}{x^3}$

$x - \frac{1}{x} = 7$ Taking cube of both sides

$$\left(x - \frac{1}{x}\right)^3 = (7)^3$$

$$x^3 - \frac{1}{x^3} - 3(x)\left(\frac{1}{x}\right)\left(x - \frac{1}{x}\right) = 343$$

$$x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right) = 343$$

$$x^3 - \frac{1}{x^3} - 3(7) = 343$$

$$x^3 - \frac{1}{x^3} - 21 = 343$$

$$x^3 - \frac{1}{x^3} = 343 + 21$$

$$x^3 - \frac{1}{x^3} = 364$$

12. If $3x + \frac{1}{3x} = 5$, then find value of $27x^3 + \frac{1}{27x^3}$

$$\left(3x + \frac{1}{3x}\right)^3 = (5)^3$$

$$(3x)^3 + \left(\frac{1}{3x}\right)^3 + 3(3x)\left(\frac{1}{3x}\right)\left(3x + \frac{1}{3x}\right) = 125$$

$$27x^3 + \frac{1}{27x^3} + 3\left(3x + \frac{1}{3x}\right) = 125$$

$$27x^3 + \frac{1}{27x^3} + 3(5) = 125$$

$$27x^3 + \frac{1}{27x^3} + 15 = 125$$

$$27x^3+\frac{1}{27x^3}=125-15$$

$$27x^3+\frac{1}{27x^3}=110$$

13. If $\left(5x-\frac{1}{5x}\right)=6$, then find value of

$$125x^3-\frac{1}{25x^3}.$$

$$\left(5x-\frac{1}{5x}\right)=6$$

Taking cube of both sides

$$\left(5x-\frac{1}{5x}\right)^3=(6)^3$$

$$(5x)^3-\left(\frac{1}{5x}\right)^3-3(\cancel{5x})\left(\frac{1}{\cancel{5x}}\right)\left(5x-\frac{1}{5x}\right)=216$$

$$125x^3-\frac{1}{125x^3}-3\left(5x-\frac{1}{5x}\right)=216$$

$$125x^3-\frac{1}{125x^3}-3(6)=216$$

$$125x^3-\frac{1}{25x^3}-18=216$$

$$125x^3-\frac{1}{125x^3}=216+18$$

$$125x^3-\frac{1}{125x^3}=234$$

14. Factorize (i) x^3-y^3-x+y

(i) x^3-y^3-x+y

$$=(x-y)(x^2+xy+y^2)-1(x-y)$$

$$=(x-y)\left[x^2+xy+y^2-1\right]$$

(ii) $8x^3-\frac{1}{27y^3}$

$$=(2x)^3-\left(\frac{1}{3y}\right)^3$$

$$=\left(2x-\frac{1}{3y}\right)\left((2x)^2+(2x)\left(\frac{1}{3y}\right)+\left(\frac{1}{3y}\right)^2\right)$$

$$=\left(2x-\frac{1}{3y}\right)\left(4x^2+\frac{2x}{3y}+\frac{1}{9y^2}\right)$$

15. Find products, using formulae

(i) $(x^2+y^2)(x^4-x^2y^2+y^4)$

$$=(x^2)^3+(y^2)^3$$

$$\text{Ref}=(a+b)(a^2-ab+b^2)=a^3+b^3$$

$$=x^6+y^6$$

(ii) $(x^3-y^3)(x^6+x^3y^3+y^6)$

$$=(x^3)^3-(y^3)^3$$

$$\text{Ref.}(a-b)(a^2+ab+b^2)=a^3-b^3$$

$$=x^9-y^9$$

(iii) $(x-y)(x+y)(x^2+y^2)(x^2+xy+y^2)$

$(x^2-xy+y^2)(x^4-x^2y^2+y^4)$

$$=(x-y)(x^2+xy+y^2)(x+y)(x^2-xy+y^2)$$

$(x^2+y^2)(x^4-x^2y^2+y^4)$

$$=(x^3-y^3)(x^3+y^3)\left[(x^2)^3+(y^2)^3\right]$$

$$=\left[(x^3)^2-(y^3)^2\right]\left(x^6+y^6\right)$$

$$=(x^6-y^6)(x^6+y^6)$$

$$=(x^6)^2-(y^6)^2$$

$$=x^{12}-y^{12}$$

16. $(2x^2-1)(2x^2+1)(4x^4+2x^2+1)$

$(4x^4-2x^2+1)$

$$=(2x^2-1)(4x^4+2x^2+1)(2x^2+1)$$

$(4x^4-2x^2+1)$

$$=((2x^2)^3-(1)^3)((2x^2)^3+(1)^3)$$

$= (8x^6 - 1)(8x^6 + 1)$

$= (8x^6)^2 - (1)^2$

$= 64x^{12} - 1$

Define Surd

An irrational radical with rational radicand is called a surd.

Hence the radical $\sqrt[n]{a}$ is a surd if

(i) a is rational

(ii) the result $\sqrt[n]{a}$ is irrational.

e.g., $\sqrt{3}, \sqrt{2/5}, \sqrt[3]{7}, \sqrt[4]{10}$ are surds.

But $\sqrt{\pi}$ is not surd because is π not rational.

Note: Every surd is an irrational number but every irrational number is not surd

Example

Simplify by combining similar terms.

(i) $4\sqrt{3} - 3\sqrt{27} + 2\sqrt{75}$

(ii) $\sqrt[3]{128} - \sqrt[3]{250} + \sqrt[3]{432}$

Solution

(i) $4\sqrt{3} - 3\sqrt{27} + 2\sqrt{75}$

$= 4\sqrt{3} - 3\sqrt{9\times3} + 2\sqrt{25\times3} = 4\sqrt{3} - 3\sqrt{9}\sqrt{3} + 2\sqrt{25}\times\sqrt{3}$

$= 4\sqrt{3} - 9\sqrt{3} + 10\sqrt{3} = (4-9+10)\sqrt{3} = 5\sqrt{3}$

(ii) $\sqrt[3]{128} - \sqrt[3]{250} + \sqrt[3]{432}$

$= \sqrt[3]{64\times2} - \sqrt[3]{125\times2} + \sqrt[3]{216\times2}$

$= \sqrt[3]{(4)^3\times2} - \sqrt[3]{(5)^3\times2} + \sqrt[3]{(6)^3\times2}$

$= \sqrt[3]{(4)^3}\,\sqrt[3]{2} - \sqrt[3]{(5)^3}\,\sqrt[3]{2} + \sqrt[3]{(6)^3}\,\sqrt[3]{2}$

$= 4\sqrt[3]{2} - 5\sqrt[3]{2} + 6\sqrt[3]{2} = (4-5+6)\sqrt[3]{2} = 5\sqrt[3]{2}$

Example

Simplify and express the answer in the simplest form.

(i) $\sqrt{14}\sqrt{35}$ (ii) $\dfrac{\sqrt[6]{12}}{\sqrt{3}\sqrt[3]{2}}$

Solution

(i) $\sqrt{14}\sqrt{35} = \sqrt{14\times35} = \sqrt{7\times2\times7\times5} = \sqrt{(7)^2\times2\times5}$

$$= \sqrt{(7)^2\times 10} = \sqrt{(7)^2}\times\sqrt{10} = 7\sqrt{10}$$

(ii) We have $\dfrac{\sqrt[6]{12}}{\sqrt{3}\,\sqrt[3]{2}}$

For $\sqrt{3}\ \sqrt[3]{2}$ the L.C.M of orders 2 and 3 is 6.

Thus $\sqrt{3} = (3)^{1/2} = (3)^{3/6} = \sqrt[6]{3^3} = \sqrt[6]{27}$

and $\sqrt[3]{2} = (2)^{1/3} = (2)^{2/6} = \sqrt[6]{(2)^2} = \sqrt[6]{4}$

Hence $\dfrac{\sqrt[6]{12}}{\sqrt{3}\,\sqrt[3]{2}} = \dfrac{\sqrt[6]{12}}{\sqrt[6]{27}\,\sqrt[6]{4}} = \dfrac{\sqrt[6]{12}}{\sqrt[6]{108}} = \sqrt[6]{\dfrac{12}{108}} = \sqrt[6]{\dfrac{1}{9}}$

Its simplest form is

$$\sqrt[6]{\left(\frac{1}{3}\right)^2} = \left(\frac{1}{3}\right)^{2/6} = \left(\frac{1}{3}\right)^{1/3} = \sqrt[3]{\frac{1}{3}}$$

Exercise 4.3

1. Express each of the following surd in the simplest form.

(i) $\sqrt{180}$

$= \sqrt{2x2x3x3x5}$

$= 2x3\sqrt{5}$

$= 6\sqrt{5}$

(ii) $3\sqrt{162}$

$= 3\sqrt{2x3x3x3x3}$

$= 3(3x3)\sqrt{2}$

$= 27\sqrt{2}$

(iii) $\dfrac{3}{4}\sqrt[3]{128}$

$= \dfrac{3}{4}(128)^{\frac{1}{3}}$

$= \dfrac{3}{4}(2x2x2x2x2x2x2)^{\frac{1}{3}}$

$= \dfrac{3}{4}\left(2^3x2^3x2\right)^{\frac{1}{3}}$

$= \dfrac{3}{4}\left(2^3\right)^{\frac{1}{3}}\times\left(2^3\right)^{\frac{1}{3}}\times 2^{\frac{1}{3}}$

$= \dfrac{3}{4}(2)(2)\times\sqrt[3]{2}$

$= 3\sqrt[3]{2}$

(iv) $\sqrt[5]{96x^6y^7z^8}$

$= \sqrt[5]{2x2x2x2x2x3x^6y^7z^8}$

$= \left(2^5x3x^5.x.y^5.y^2.z^5.z^3\right)^{\frac{1}{5}}$

$= (2^5)^{\frac{1}{5}}(3)^{\frac{1}{5}}(x^5)^{\frac{1}{5}}.x^{\frac{1}{5}}.(y^5)^{\frac{1}{5}}.(y^2)^{\frac{1}{5}}.(z^5)^{\frac{1}{5}}(z^3)^{\frac{1}{5}}$

$= 2\ 3^{\frac{1}{5}}.x.x^{\frac{1}{5}}y.y^{\frac{2}{5}}.z.z^{\frac{3}{5}}$

$= 2xyz\ 3^{\frac{1}{5}}.x^{\frac{1}{5}}.y^{\frac{2}{5}}.z^{\frac{3}{5}}$

$= 2xyz\sqrt[5]{3xy^2z^3}$

2. Simplify

(i) $\dfrac{\sqrt{18}}{\sqrt{3}.\sqrt{2}} = \dfrac{\sqrt{3.3.2}}{\sqrt{3}.\sqrt{2}} = \dfrac{3\cancel{\sqrt{2}}}{\sqrt{3}.\cancel{\sqrt{2}}}$

$= \dfrac{3}{\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{\cancel{3}\sqrt{3}}{\cancel{3}} = \sqrt{3}$

(ii) $\dfrac{\sqrt{21}\times\sqrt{9}}{\sqrt{63}} = \dfrac{\sqrt{3x7}\times\sqrt{3x3}}{\sqrt{3x3x7}}$

$= \dfrac{\sqrt{3\times7\times3\times3}}{\sqrt{3\times3\times7}}$

$= \dfrac{\cancel{3}\sqrt{21}}{\cancel{3}\sqrt{7}} = \sqrt{\dfrac{21}{7}}$

$= \sqrt{3}$

(iii) $\sqrt[5]{243x^5y^{10}z^{15}}$

$= \left(3^5.x^5y^{10}z^{15}\right)^{\frac{1}{5}}$

$= (3^5)^{\frac{1}{5}}\,(x^5)^{\frac{1}{5}}\,(y^{10})^{\frac{1}{5}}\,(z^{15})^{\frac{1}{5}}$

$= 3xy^2z^3$

(iv) $\dfrac{4}{5}\sqrt[3]{125}$

$= \dfrac{4}{\cancel{5}}\left(\cancel{5}^{\cancel{3}}\right)^{\frac{1}{\cancel{3}}}$

$= 4$

(v) $\sqrt{21}\times\sqrt{7}\times\sqrt{3}$

$= \sqrt{3x7}\times\sqrt{7}\times\sqrt{3}$

$= \sqrt{3x7x7x3} = \left(3^2x7^2\right)^{\frac{1}{2}}$

$= (3^2)^{\frac{1}{2}}x(7^2)^{\frac{1}{2}}$

$= 3 \text{ x } 7$

$= 21$

3. Simplify by combining similar terms:

(i) $\sqrt{45} - 3\sqrt{20} + 4\sqrt{5}$

$= \sqrt{9x5} - 3\sqrt{4x5} + 4\sqrt{5}$

$= 3\sqrt{5} - 6\sqrt{5} + 4\sqrt{5}$

$= (3-6+4)\sqrt{5}$

$= (-3+4)\sqrt{5}$

$= \sqrt{5}$

(ii) $4\sqrt{12} + 5\sqrt{27} - 3\sqrt{75} + \sqrt{300}$

$= 4\sqrt{3x4} + 5\sqrt{3x3x3} - 3\sqrt{3x5x5}$
$+\sqrt{3x2x5x2x5}$

$= 8\sqrt{3} + 15\sqrt{3} - 15\sqrt{3} + 10\sqrt{3}$

$= (8 + \cancel{15} - \cancel{15} + 10)\sqrt{3}$

$= 18\sqrt{3}$

(iii) $\sqrt{3}\,(2\sqrt{3} + 3\sqrt{3})$

$= \sqrt{3}\left((2+3)\sqrt{3}\right)$

$= \sqrt{3}\left(5\sqrt{3}\right)$

$= 5\sqrt{3}x\sqrt{3}$

$= 5(\sqrt{3x3})$

$= 5(3)$

$= 15$

(iv) $2(6\sqrt{5} - 3\sqrt{5})$

$= 2\left((6-3)\sqrt{5}\right)$

$= 2(3\sqrt{5})$

$= 6\sqrt{5}$

4. Simplify:

(i) $(3+\sqrt{3})(3-\sqrt{3})$

$= (3)^2 - (\sqrt{3})^2$

$=9-3$
$=6$

(ii) $\left(\sqrt{5}+\sqrt{3}\right)^2$

$=\left(\sqrt{5}\right)^2+\left(\sqrt{3}\right)^2+2\sqrt{5}\sqrt{3}$

$=5+3+2\sqrt{15}$

$=8+2\sqrt{15}$

(iii) $\left(\sqrt{5}+\sqrt{3}\right)\left(\sqrt{5}-\sqrt{3}\right)$

$=\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2$

$=5-3$
$=2$

(iv) $\left(\sqrt{2}+\frac{1}{\sqrt{3}}\right)\left(\sqrt{2}-\frac{1}{\sqrt{3}}\right)$

$=\left(\sqrt{2}\right)^2-\left(\frac{1}{\sqrt{3}}\right)^2$

$=2-\frac{1}{3}$

$=\frac{6-1}{3}=\frac{5}{3}$

(v) $\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{x}-\sqrt{y}\right)(x+y)\left(x^2+y^2\right)$

$=\left(\left(\sqrt{x}\right)^2-\left(\sqrt{y}\right)^2\right)\left((x+y)(x^2+y^2\right)$

$=(x-y)(x+y)(x^2+y^2)$

$=(x^2-y^2)(x^2+y^2)$

$=(x^2)^2-(y^2)^2$

$=x^4-y^4$

Define monomial surd

(i) A surd which contains a single term is called a monomial surd. e.g., $\sqrt{2},\sqrt{3}$ etc.

(ii) A surd which contains sum of two monomial surds or sum of a monomial surd and a rational number is called a binomial surd.
e.g., $\sqrt{3}+\sqrt{7}$ or $\sqrt{2}+5$ $\sqrt{11}-8$ etc.

(iii) If the product of two surds is a rational number, then each surd is called the rationalizing factor of the other.

Example

Rationalize the denominator $\frac{58}{7-2\sqrt{5}}$

Solution

To rationalize the denominator, we multiply both the numerator and denominator by the conjugate $(7+2\sqrt{5})$ of $(7-2\sqrt{5})$, i.e.

$$\frac{58}{7-2\sqrt{5}} = \frac{58}{7-2\sqrt{5}}\times\frac{7+2\sqrt{5}}{7+2\sqrt{5}} = \frac{58(7+2\sqrt{5})}{(7)^2-(2\sqrt{5})^2}$$

$$= \frac{58(7+2\sqrt{5})}{49-20}; \text{ (radical is eliminated in the denominator)}$$

$$= \frac{58(7+2\sqrt{5})}{29} = 2(7+2\sqrt{5})$$

Example

Rationalize the denominator $\dfrac{2}{\sqrt{5}+\sqrt{2}}$

Solution

Multiplying both the numerator and denominator by the conjugate $(\sqrt{5}-\sqrt{2})$ of $(\sqrt{5}+\sqrt{2})$, to get

$$\frac{2}{\sqrt{5}+\sqrt{2}} = \frac{2}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{2(\sqrt{5}-\sqrt{2})}{(\sqrt{5})^2-(\sqrt{2})^2} = \frac{2(\sqrt{5}-\sqrt{2})}{5-2}$$

$$= \frac{2(\sqrt{5}-\sqrt{2})}{3} = \frac{2(\sqrt{5}-\sqrt{2})}{3}$$

Example

Simplify $\dfrac{6}{2\sqrt{3}-\sqrt{6}} + \dfrac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} - \dfrac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}$

Solution

First we shall rationalize the denominators and then simplify. We have

$$\frac{6}{2\sqrt{3}-\sqrt{6}} + \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} - \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}$$

$$= \frac{6}{2\sqrt{3}-\sqrt{6}} \times \frac{2\sqrt{3}+\sqrt{6}}{2\sqrt{3}+\sqrt{6}} + \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} - \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}} \times \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}$$

$$= \frac{6(2\sqrt{3}+\sqrt{6})}{(2\sqrt{3})^2-(\sqrt{6})^2} + \frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{(\sqrt{3})^2-(\sqrt{2})^2} - \frac{4\sqrt{3}(\sqrt{6}+\sqrt{2})}{(\sqrt{6})^2-(\sqrt{2})^2}$$

$$= \frac{6(2\sqrt{3}+\sqrt{6})}{12-6} + \frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{3-2} - \frac{4\sqrt{3}(\sqrt{6}+\sqrt{2})}{6-2}$$

$$= \frac{12\sqrt{3}+6\sqrt{6})}{6} + \frac{\sqrt{6}\sqrt{3}-\sqrt{6}\sqrt{2})}{1} - \frac{4\sqrt{3}\sqrt{6}+4\sqrt{3}\sqrt{2}}{4}$$

$$= 2\sqrt{3}+\sqrt{6}+3\sqrt{2}-2\sqrt{3}-3\sqrt{2}-\sqrt{6} = 0$$

Example

Find rational numbers x and y such that $\frac{4+3\sqrt{5}}{4-3\sqrt{5}}=x+y\sqrt{5}$

Solution

$$\frac{4+3\sqrt{5}}{4-3\sqrt{5}} = \frac{4+3\sqrt{5}}{4-3\sqrt{5}}\times\frac{4+3\sqrt{5}}{4+3\sqrt{5}} = \frac{(4+3\sqrt{5})^2}{(4)^2-(3\sqrt{5})^2}$$

$$= \frac{16+24\sqrt{5}+45}{16-45} = \frac{61+24\sqrt{5}}{-29}$$

$$\Rightarrow \quad \frac{-61}{29}-\frac{24}{29}\sqrt{5} = x+y\sqrt{5} \quad \text{(given)}$$

Hence, on comparing the two sides, we get

$$x = \frac{-61}{29}, \qquad y = \frac{-24}{29}$$

Example

If $x=3+\sqrt{8}$, then evaluate

(i) $x+\frac{1}{x}$ and (ii) $x^2+\frac{1}{x^2}$

Solution

Since $x = 3+\sqrt{8}$, therefore,

$$\frac{1}{x} = \frac{1}{3+\sqrt{8}} = \frac{1}{3+\sqrt{8}}\times\frac{3-\sqrt{8}}{3-\sqrt{8}} = \frac{3-\sqrt{8}}{(3)^2-(\sqrt{8})^2}$$

$$= \frac{3-\sqrt{8}}{9-8} = 3-\sqrt{8}$$

(i) $x+\frac{1}{x} = 3+\sqrt{8}+3-\sqrt{8}=6$

(ii) $\left(x+\frac{1}{x}\right)^2 = 36$

or $x^2+2x\times\frac{1}{x}+\frac{1}{x^2}=36$

or $x^2+\frac{1}{x^2}=34$

Exercise 4.4

1. Rationalize the denominator

(i) $$\frac{3}{4\sqrt{3}}=\frac{3}{4\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{3\sqrt{3}}{4\sqrt{3x3}}$$

$$=\frac{3\sqrt{3}}{4(3)}=\frac{\sqrt{3}}{4}$$

(ii) $$\frac{14}{\sqrt{98}}=\frac{14}{7\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$

$$=\frac{\cancel{14}\sqrt{2}}{\cancel{14}}=\sqrt{2}$$

(iii) $$\frac{6}{\sqrt{8}.\sqrt{27}}=\frac{6}{2\sqrt{2}.3\sqrt{3}}$$

$$=\frac{\cancel{6}}{\cancel{6}\sqrt{6}}$$

$$=\frac{1}{\sqrt{6}}$$

$$=\frac{1}{\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}$$

$$=\frac{\sqrt{6}}{6}$$

(iv) $$\frac{1}{3+2\sqrt{5}}=\frac{1}{3+2\sqrt{5}}\times\frac{3-2\sqrt{5}}{3-2\sqrt{5}}$$

$$=\frac{3-2\sqrt{5}}{(3)^2-(2\sqrt{5})^2}=\frac{3-2\sqrt{5}}{9-20}$$

$$=\frac{3-2\sqrt{5}}{-11}$$

(v) $$\frac{15}{\sqrt{31}-4}$$

$$=\frac{15}{\sqrt{31}-4}\times\frac{\sqrt{31}+4}{\sqrt{31}+4}$$

$$=\frac{15\left(\sqrt{31}+4\right)}{\left(\sqrt{31}\right)^2-(4)^2}$$

$$=\frac{15\left(\sqrt{31}+4\right)}{31-16}$$

$$=\frac{\cancel{15}\left(\sqrt{31}+4\right)}{\cancel{15}}$$

$$=\sqrt{31}+4$$

(vi) $$\frac{2}{\sqrt{5}-\sqrt{3}}=\frac{2}{\sqrt{5}-\sqrt{3}}\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

$$=\frac{2(\sqrt{5}+\sqrt{3})}{(\sqrt{5})^2-(\sqrt{3})^2}$$

$$=\frac{2\left(\sqrt{5}+\sqrt{3}\right)}{5-3}$$

$$=\frac{\cancel{2}\left(\sqrt{5}+\sqrt{3}\right)}{\cancel{2}}$$

$$=\sqrt{5}+\sqrt{3}$$

(vii) $$\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{\sqrt{3}-1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$=\frac{\left(\sqrt{3}-1\right)\left(\sqrt{3}-1\right)}{\left(\sqrt{3}\right)^2-(1)^2}$$

$$=\frac{\left(\sqrt{3}-1\right)^2}{3-1}$$

$$=\frac{\left(\sqrt{3}\right)^2+1^2-2(1)\sqrt{3}}{2}$$

$$=\frac{3+1-2\sqrt{3}}{2}$$

$= \frac{4-2\sqrt{3}}{2}$

$= \frac{2(2-\sqrt{3})}{2}$

$= 2-\sqrt{3}$

(viii) $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} \times \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$

$= \frac{(\sqrt{5}+\sqrt{3})^2}{(\sqrt{5})^2-(\sqrt{3})^2}$

$= \frac{(\sqrt{5}+\sqrt{3})^2}{5-3}$

$= \frac{(\sqrt{5})^2+(\sqrt{3})^2+2(\sqrt{5})(\sqrt{3})}{2}$

$= \frac{5+3+2\sqrt{15}}{2}$

$= \frac{8+2\sqrt{15}}{2}$

$= \frac{2(4+\sqrt{15})}{2}$

$= 4+\sqrt{15}$

(2) Find conjugate of $x+\sqrt{y}$:

(i) $3+\sqrt{7}$

Conjugate of $3+\sqrt{7}$ is $3-\sqrt{7}$

(ii) $4-\sqrt{5}$

Conjugate of $4-\sqrt{5}$ is $4+\sqrt{5}$

(iii) $2+\sqrt{3}$

Conjugate of $2+\sqrt{3}$ is $2-\sqrt{3}$

(iv) $2+\sqrt{5}$

Conjugate of $2+\sqrt{5}$ is $2-\sqrt{5}$

(v) $5+\sqrt{7}$

Conjugate of $5+\sqrt{7}$ is $5-\sqrt{7}$

(vi) $4-\sqrt{15}$

Conjugate of $4-\sqrt{15}$ is $4+\sqrt{15}$

(vii) $7-\sqrt{6}$

Conjugate of $7-\sqrt{6}$ is $7+\sqrt{6}$

(viii) $9+\sqrt{2}$

Conjugate of $9+\sqrt{2}$ is $9-\sqrt{2}$

Q.3 If $x=2-\sqrt{3}$ find $\frac{1}{x}$

(i) $x = 2-\sqrt{3}$

$\frac{1}{x} = \frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$

$\frac{1}{x} = \frac{2+\sqrt{3}}{(2)^2-(\sqrt{3})^2}$

$\frac{1}{x} = \frac{2+\sqrt{3}}{4-3}$

$\frac{1}{x} = 2+\sqrt{3}$

(ii) $x=4-\sqrt{17}$ **find** $\frac{1}{x}$

$\frac{1}{x} = \frac{1}{4-\sqrt{17}} \times \frac{4+\sqrt{17}}{4+\sqrt{17}}$

$\frac{1}{x} = \frac{4+\sqrt{17}}{(4)^2-(\sqrt{17})^2}$

$= \frac{4+\sqrt{17}}{16-17}$

$= \frac{4+\sqrt{17}}{-1}$

$= -(4+\sqrt{17})$

$= -4-\sqrt{17}$

(iii) If $x=\sqrt{3}+2$, **find** $x+\dfrac{1}{x}$

$$x=\sqrt{3}+2$$

$$\frac{1}{x}=\frac{1}{\sqrt{3}+2}\times\frac{\sqrt{3}-2}{\sqrt{3}-2}$$

$$\frac{1}{x}=\frac{\sqrt{3}-2}{\left(\sqrt{3}\right)^2-(2)^2}$$

$$\frac{1}{x}=\frac{\sqrt{3}-2}{3-4}$$

$$\frac{1}{x}=\frac{\sqrt{3}-2}{-1}$$

$$\frac{1}{x}=-\sqrt{3}+2=2-\sqrt{3}$$

$$x+\frac{1}{x}=\cancel{\sqrt{3}}+2-\cancel{\sqrt{3}}+2$$

$$x+\frac{1}{x}=4$$

Q4. Simplify

(i) $\dfrac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}}+\dfrac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$

$$\frac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}}\times\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}+\frac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

$$=\frac{\left(1+\sqrt{2}\right)\left(\sqrt{5}-\sqrt{3}\right)}{\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2}+\frac{\left(1-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2}$$

$$=\frac{\left(1+\sqrt{2}\right)\left(\sqrt{5}-\sqrt{3}\right)}{5-3}+\frac{\left(1-\sqrt{2}\right)\left(\sqrt{5}+\sqrt{3}\right)}{5-3}$$

$$=\frac{\sqrt{5}-\sqrt{3}+\sqrt{2}\sqrt{5}-\sqrt{2}\sqrt{3}}{2}+\frac{\sqrt{5}+\sqrt{3}-\sqrt{2}\sqrt{5}-\sqrt{2}\sqrt{3}}{2}$$

$$=\frac{\sqrt{5}-\sqrt{3}+\sqrt{10}-\sqrt{6}}{2}+\frac{\sqrt{5}+\sqrt{3}-\sqrt{10}-\sqrt{6}}{2}$$

$$=\frac{\sqrt{5}-\cancel{\sqrt{3}}+\cancel{\sqrt{10}}-\sqrt{6}+\sqrt{5}+\cancel{\sqrt{3}}-\cancel{\sqrt{10}}-\sqrt{6}}{2}$$

$$=\frac{2\sqrt{5}-2\sqrt{6}}{2}$$

$$=\frac{\cancel{2}\left(\sqrt{5}-\sqrt{6}\right)}{\cancel{2}}$$

$$=\sqrt{5}-\sqrt{6}$$

(ii) $\dfrac{1}{2+\sqrt{3}}+\dfrac{2}{\sqrt{5}-\sqrt{3}}+\dfrac{1}{2+\sqrt{5}}$

$$=\frac{1}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}+\frac{2}{\sqrt{5}-\sqrt{3}}$$

$$\times\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}+\frac{1}{2+\sqrt{5}}\times\frac{2-\sqrt{5}}{2-\sqrt{5}}$$

$$=\frac{2-\sqrt{3}}{(2)^2-\left(\sqrt{3}\right)^2}+\frac{2\left(\sqrt{5}+\sqrt{3}\right)}{\left(\sqrt{5}\right)^2-\left(\sqrt{3}\right)^2}+\frac{2-\sqrt{5}}{(2)^2-\left(\sqrt{5}\right)^2}$$

$$=\frac{2-\sqrt{3}}{4-3}+\frac{2\left(\sqrt{5}+\sqrt{3}\right)}{5-3}+\frac{2-\sqrt{5}}{4-5}$$

$$=2-\sqrt{3}+\frac{2\left(\sqrt{5}+\sqrt{3}\right)}{2}+\frac{2-\sqrt{5}}{-1}$$

$$=\cancel{2}-\cancel{\sqrt{3}}+\sqrt{5}+\cancel{\sqrt{3}}-\cancel{2}+\sqrt{5}=2\sqrt{5}$$

(iii) $\dfrac{2}{\sqrt{5}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{2}}-\dfrac{3}{\sqrt{5}+\sqrt{2}}$

$$=\frac{2}{\sqrt{5}+\sqrt{3}}\times\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{2}}$$

$$\times\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}-\frac{3}{\sqrt{5}+\sqrt{2}}\times\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$$

$$=\frac{2\left(\sqrt{5}-\sqrt{3}\right)}{\left(\sqrt{5}\right)^2-\left(\sqrt{5}\right)^2}+\frac{\sqrt{3}-\sqrt{2}}{\left(\sqrt{3}\right)^2-\left(\sqrt{2}\right)^2}-\frac{3\left(\sqrt{5}-\sqrt{2}\right)}{\left(\sqrt{5}\right)^2-\left(\sqrt{2}\right)^2}$$

$$=\frac{2\left(\sqrt{5}-\sqrt{3}\right)}{5-3}+\frac{\sqrt{3}-\sqrt{2}}{3-2}-\frac{3\left(\sqrt{5}-\sqrt{2}\right)}{5-2}$$

$$=\frac{\cancel{2}\left(\sqrt{5}-\sqrt{3}\right)}{\cancel{2}}+\frac{\sqrt{3}-\sqrt{2}}{1}-\frac{\cancel{3}\left(\sqrt{5}-\sqrt{2}\right)}{\cancel{3}}$$

$$=\cancel{\sqrt{5}}-\cancel{\sqrt{3}}+\cancel{\sqrt{3}}-\cancel{\sqrt{2}}-\cancel{\sqrt{5}}+\cancel{\sqrt{2}}$$

$= 0$

Q5(i) If $x=2+\sqrt{3}$, find value of $x-\frac{1}{x}$ and $\left(x-\frac{1}{x}\right)^2$

$$x=2+\sqrt{3}$$

$$\frac{1}{x}=\frac{1}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}$$

$$\frac{1}{x}=\frac{2-\sqrt{3}}{(2)^2-\left(\sqrt{3}\right)^2}$$

$$\frac{1}{x}=2-\sqrt{3}$$

$$x-\frac{1}{x}=2+\sqrt{3}-\left(2-\sqrt{3}\right)$$

$$=\cancel{2}+\sqrt{3}-\cancel{2}+\sqrt{3}$$

$$=2\sqrt{3}$$

$$\left(x-\frac{1}{x}\right)^2=\left(2\sqrt{3}\right)^2$$

$$\left(x-\frac{1}{x}\right)^2=12$$

(ii) If $x=\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$ find the value of $x+\frac{1}{x}$, $x^2+\frac{1}{x}$ and $x^3+\frac{1}{x^3}$

$$x=\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}\times\frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$$

$$x=\frac{\left(\sqrt{5}-\sqrt{2}\right)^2}{\left(\sqrt{5}\right)^2-\left(\sqrt{2}\right)^2}$$

$$x=\frac{\left(\sqrt{5}\right)^2+\left(\sqrt{2}\right)^2-2\left(\sqrt{5}\right)\left(\sqrt{2}\right)}{5-2}$$

$$x=\frac{5+2-2\sqrt{10}}{3}$$

$$x=\frac{7-2\sqrt{10}}{3}$$

$$\frac{1}{x}=\frac{3}{7-2\sqrt{10}}\times\frac{7+2\sqrt{10}}{7+2\sqrt{10}}$$

$$\frac{1}{x}=\frac{3\left(7+2\sqrt{10}\right)}{(7)^2-\left(2\sqrt{10}\right)^2}$$

$$\frac{1}{x}=\frac{3\left(7+2\sqrt{10}\right)}{49-40}$$

$$\frac{1}{x}=\frac{3\left(7+2\sqrt{10}\right)}{9}$$

$$\frac{1}{x}=\frac{7+2\sqrt{10}}{3}$$

$$x+\frac{1}{x}=\frac{7-2\sqrt{10}}{3}+\frac{7+2\sqrt{10}}{3}$$

$$=\frac{7-\cancel{2\sqrt{10}}+7+\cancel{2\sqrt{10}}}{3}=\frac{14}{3}$$

Now

$$x+\frac{1}{x}=\frac{14}{3}$$

Squaring

$$\left(x+\frac{1}{x}\right)^2=\left(\frac{14}{3}\right)^2$$

$$x^2+\frac{1}{x^2}+2=\frac{196}{9}$$

$$x^2+\frac{1}{x^2}=\frac{196}{9}-2$$

$$x^2+\frac{1}{x^2}=\frac{196-18}{9}=\frac{178}{9}$$

Also

$$x^3+\frac{1}{x^3}=?$$

$$x+\frac{1}{x}=\frac{14}{3}$$

$$\left(x+\frac{1}{x}\right)^3=\left(\frac{14}{3}\right)^3$$

$$x^3+\frac{1}{x^3}+3(x)\left(\frac{1}{x}\right)\left(x+\frac{1}{x}\right)=\frac{2744}{27}$$

$$x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)=\frac{2744}{27}$$

$$x^3+\frac{1}{x^3}+\cancel{3}\left(\frac{14}{\cancel{3}}\right)=\frac{2744}{27}$$

$$x^3+\frac{1}{x^3}=\frac{2744}{27}-14$$

$$=\frac{2366}{27}$$

Q6. Determine the rational numbers a and b. If

$$\frac{\sqrt{3}-1}{\sqrt{3}+1}+\frac{\sqrt{3}+1}{\sqrt{3}-1}=a+b\sqrt{3}$$

Given

$$\frac{\sqrt{3}-1}{\sqrt{3}+1}+\frac{\sqrt{3}+1}{\sqrt{3}-1}=a+b\sqrt{3}$$

$$\frac{\sqrt{3}-1}{\sqrt{3}+1}\times\frac{\sqrt{3}-1}{\sqrt{3}-1}+\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}=a+b\sqrt{3}$$

$$\frac{\left(\sqrt{3}-1\right)^2}{\left(\sqrt{3}\right)^2-(1)^2}+\frac{\left(\sqrt{3}+1\right)^2}{\left(\sqrt{3}\right)^2-(1)^2}=a+b\sqrt{3}$$

$$\frac{\left(\sqrt{3}\right)^2+(1)^2-2\left(\sqrt{3}\right)(1)}{3-1}+\frac{\left(\sqrt{3}\right)^2+(1)^2+2\sqrt{3}}{3-1}=a+b\sqrt{3}$$

$$\frac{3+1-2\sqrt{3}}{2}+\frac{3+1+2\sqrt{3}}{2}=a+b\sqrt{3}$$

$$\frac{4-2\sqrt{3}}{2}+\frac{4+2\sqrt{3}}{2}=a+b\sqrt{3}$$

$$\frac{\cancel{2}\left(2-\sqrt{3}\right)}{\cancel{2}}+\frac{\cancel{2}\left(2+\sqrt{3}\right)}{\cancel{2}}=a+b\sqrt{3}$$

$$2-\cancel{\sqrt{3}}+2+\cancel{\sqrt{3}}=a+b\sqrt{3}$$

$$4=a+b\sqrt{3}$$

$$\Rightarrow\ a+b\sqrt{3}=4$$

Hence on comparing the two sides, we get

$\Rightarrow\ a=4$ and $b=0$

Exercise

Q1. If $x+\frac{1}{x}=3$ find

(i) $x^2+\frac{1}{x^2}$ **(ii)** $x^4+\frac{1}{x^4}$

(i) $x+\frac{1}{x}=3$

$$\left(x+\frac{1}{x}\right)^2=(3)^2$$

$$x^2+\frac{1}{x^2}+2=9$$

$$x^2+\frac{1}{x^2}=9-2$$

$$x^2+\frac{1}{x^2}=7$$

(ii) $x^4+\frac{1}{x^4}$

$$\left(x^2+\frac{1}{x^2}\right)^2=(7)^2$$

$$x^4+\frac{1}{x^4}+2=49$$

$$x^4+\frac{1}{x^4}=49-2$$

$$x^4+\frac{1}{x^4}=47$$

Q2. If $x-\frac{1}{x}=2$ find

(i) $x^2+\frac{1}{x^2}$ **(ii)** $x^4+\frac{1}{x^4}$

(i) $x-\frac{1}{x}=2$

Squaring

$$\left(x-\frac{1}{x}\right)^2=(2)^2$$

$$x^2+\frac{1}{x^2}-2=4$$

$$x^2+\frac{1}{x^2}=4+2$$

$$x^2+\frac{1}{x^2}=6$$

(ii) $\left(x^2+\frac{1}{x^2}\right)=(6)^2$

$$x^4+\frac{1}{x^4}+2=36$$

$$x^4+\frac{1}{x^4}=36-2$$

$$x^4+\frac{1}{x^4}=34$$

Q3. Find value of x^3+y^3 and xy if $x+y=5$ and $x-y=3$

$$4xy=(x+y)^2-(x-y)^2$$

$$4xy=(5)^2-(3)^2$$

Now

$4xy=25-9=16$

$xy=\frac{16}{4}=4$

$$x+y=5$$

taking cube both sides

$$(x+y)^3=(5)^3$$

$$x^3+y^3+3xy(x+y)=125$$

$$x^3+y^3+3(4)(5)=125$$

$$x^3+y^3+60=125$$

$$x^3+y^3=125-60$$

$$x^3+y^3=65$$

Q4. If $P=2+\sqrt{3}$ find (i) $P+\frac{1}{P}$

(ii) $P-\frac{1}{P}$ **(iii)** $P^2+\frac{1}{P^2}$ **(iv)** $P^2-\frac{1}{P^2}$

$$P=2+\sqrt{3}$$

$$\frac{1}{P}=\frac{1}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}}$$

$$\frac{1}{P}=\frac{2-\sqrt{3}}{(2)^2-(\sqrt{3})^2}=\frac{2-\sqrt{3}}{4-3}=2-\sqrt{3}$$

i) $P+\frac{1}{P}=2+\cancel{\sqrt{3}}+2-\cancel{\sqrt{3}}=4$

ii) $P-\frac{1}{P}=\cancel{2}+\sqrt{3}-\cancel{2}+\sqrt{3}=2\sqrt{3}$

iii) $P^2+\frac{1}{P^2}=?$

$$\left(P+\frac{1}{P}\right)^2=(4)^2$$

$$P^2+\frac{1}{P^2}+2=16$$

$$P^2+\frac{1}{P^2}=16-2$$

$$P^2+\frac{1}{P^2}=14$$

iv) $P^2-\frac{1}{P^2}=?$

$$P^2-\frac{1}{P^2}=\left(P+\frac{1}{P}\right)\left(P-\frac{1}{P}\right)$$

$$=(4)\left(\sqrt{3}\right)$$

$$=8\sqrt{3}$$

Q5. If $q=\sqrt{5}+2$ Find (i) $q+\frac{1}{q}$

(ii) $q-\frac{1}{q}$ (iii) $q^2+\frac{1}{q^2}$ (iv) $q^2-\frac{1}{q^2}$

Solution: $q=\sqrt{5}+2$

$$\frac{1}{q}=\frac{1}{\sqrt{5}+2}\times\frac{\sqrt{5}-2}{\sqrt{5}-2}$$

$$\frac{1}{q}=\frac{\sqrt{5}-2}{\left(\sqrt{5}\right)^2-(2)^2}$$

$$\frac{1}{q}=\frac{\sqrt{5}-2}{1}=\sqrt{5}-2$$

(i) $q+\frac{1}{q}=\sqrt{5}+2+\sqrt{5}-2$

$$=2\sqrt{5}$$

(ii) $q-\frac{1}{q}=\sqrt{5}+2-\sqrt{5}+2$

$$=4$$

(iii) $q^2+\frac{1}{q^2}$

$$\left(q+\frac{1}{q}\right)^2=\left(2\sqrt{5}\right)^2$$

$$q^2+\frac{1}{q^2}+2=20$$

$$q^2+\frac{1}{q^2}=20-2$$

$$q^2+\frac{1}{q^2}=18$$

(iv) $q^2-\frac{1}{q^2}=\left(q+\frac{1}{q}\right)\left(q-\frac{1}{q}\right)$

$$=\left(2\sqrt{5}\right)(4)$$

$$=8\sqrt{5}$$

Q6. Simplify

i) $\dfrac{\sqrt{a^2+2}+\sqrt{a^2-2}}{\sqrt{a^2+2}-\sqrt{a^2-2}}$

$$=\frac{\sqrt{a^2+2}+\sqrt{a^2-2}}{\sqrt{a^2+2}-\sqrt{a^2-2}}\times\frac{\sqrt{a^2+2}+\sqrt{a^2-2}}{\sqrt{a^2+2}+\sqrt{a^2-2}}$$

$$=\frac{\left(\sqrt{a^2+2}+\sqrt{a^2-2}\right)^2}{\left(\sqrt{a^2+2}\right)^{\cancel{2}}-\left(\sqrt{a^2-2}\right)^{\cancel{2}}}$$

$$=\frac{\left(\sqrt{a^2+2}\right)^{\cancel{2}}+\left(\sqrt{a^2-2}\right)^{\cancel{2}}+2\left(\sqrt{a^2+2}\right)\left(\sqrt{a^2-2}\right)}{\cancel{a^2}+2-\cancel{a^2}+2}$$

$$=\frac{a^2+\cancel{2}+a^2-\cancel{2}+2\sqrt{a^4-4}}{4}$$

$$=\frac{2a^2+2\sqrt{a^4-4}}{4}$$

$$=\frac{\cancel{2}\left(a^2+\sqrt{a^4-4}\right)}{\cancel{4}}$$

$$=\frac{a^2+\sqrt{a^4-4}}{2}$$

(ii) $\dfrac{1}{a-\sqrt{a^2-x^2}}-\dfrac{1}{a+\sqrt{a^2-x^2}}$

$$=\frac{1}{a-\sqrt{a^2-x^2}}\times\frac{a+\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}}$$

$$-\frac{1}{a+\sqrt{a^2-x^2}}\times\frac{a-\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}}$$

$$=\frac{a+\sqrt{a^2-x^2}}{(a)^2-\left(\sqrt{a^2-x^2}\right)^2}-\frac{a-\sqrt{a^2-x^2}}{(a)^2-\left(\sqrt{a^2-x^2}\right)^2}$$

$$= \frac{a+\sqrt{a^2-x^2}}{\cancel{a^2}-\cancel{a^2}+x^2} - \frac{a-\sqrt{a^2-x^2}}{\cancel{a^2}-\cancel{a^2}+x^2}$$

$$= \frac{a+\sqrt{a^2-x^2}}{x^2} - \frac{a-\sqrt{a^2-x^2}}{x^2}$$

$$= \frac{\cancel{a}+\sqrt{a^2-x^2}-\cancel{a}+\sqrt{a^2-x^2}}{x^2}$$

$$= \frac{2\sqrt{a^2-x^2}}{x^2}$$

Objective

1. $4x + 3y - 2$ is an algebraic____
- (a) Expression
- (b) Sentence
- (c) Equation
- (d) In equation

2. The degree of polynomial $4x^4+2x^2y$ is ____
- (a) 1 (b) 2
- (c) 3 (d) 4

3. $a^3 + b^3$ is equal to____
- (a) $(a-b)(a^2+ab+b^2)$
- (b) $(a+b)(a^2-ab+b^2)$
- (c) $(a-b)(a^2-ab+b^2)$
- (d) $(a-b)(a^2+ab-b^2)$

4. $(3+\sqrt{2})(3-\sqrt{2})$ is equal to:____
- (a) 7 (b) -7
- (c) -1 (d) 1

5. Conjugate of Surd $a+\sqrt{b}$ is____
- (a) $-a+\sqrt{b}$ (b) $a-\sqrt{b}$
- (d) $\sqrt{a}+\sqrt{b}$ (d) $\sqrt{a}-\sqrt{b}$

6. $\frac{1}{a-b} - \frac{1}{a+b}$ is equal to
- (a) $\frac{2a}{a^2-b^2}$ (b) $\frac{2b}{a^2-b^2}$
- (c) $\frac{-2a}{a^2-b^2}$ (d) $\frac{-2b}{a^2-b^2}$

7. $\frac{a^2-b^2}{a+b}$ is equal to:
- (a) $(a-b)^2$ (b) $(a+b)^2$
- (c) $a+b$ (d) $a-b$

8. $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$ is equal to:____
- (a) a^2+b^2 (b) a^2-b^2
- (c) $a-b$ (d) $a+b$

9. The degree of the polynomial $x^2y^2+3xy+y^3$ is ____
- (a) 4 (b) 5
- (c) 6 (d) 2

10. $x^2-4 =$ ____
- (a) $(x-2)(x+2)$ (b) $(x-2)(x-2)$
- (c) $(x+2)(x+2)$ (d) None

11. $x^3+\frac{1}{x^3} = \left(x+\frac{1}{x}\right)(........)$
- (a) $x^2-1+\frac{1}{x^2}$ (b) $x^2+1+\frac{1}{x^2}$
- (c) $x^2+1-\frac{1}{x^2}$ (d) $x^2-1-\frac{1}{x^2}$

12. $2(a^2+b^2) =$ ____
- (a) $(a+b)^2+(a-b)^2$
- (b) $(a+b)^2-(a-b)^2$
- (c) $(a+b)^2$ (d) 4ab

13. Order of surd $\sqrt[3]{x}$ is ____
- (a) 3 (b) $\frac{1}{3}$
- (c) 0 (d) 1

14. $\dfrac{1}{2-\sqrt{3}} =$ ____

(a) $2+\sqrt{3}$ (b) $2-\sqrt{3}$

(d) $-2+\sqrt{3}$ (d) $-2-\sqrt{3}$

15. $(a+b)^2 - (a-b)^2 =$ ___

(a) $2(a^2+b^2)$ (b) 4ab

(c) 2ab (d) 3ab

16. $\sqrt{14}.\sqrt{35} =$ _____

(a) $\sqrt[4]{10}$ (b) $\sqrt[5]{10}$

(c) $7\sqrt{10}$ (d) $8\sqrt{10}$

17. A surd which contains a single term is called surd.

(a) Monomial

(b) Binomial

(c) Trinomial

(d) None

ANSWER KEY

1.	a	**2.**	d	**3.**	b	**4.**	a	**5.**	b
6.	b	**7.**	d	**8.**	c	**9.**	a	**10.**	a
11.	a	**12.**	a	**13.**	a	**14.**	a	**15.**	b
16.	c	**17.**	a						

Unit 05

FACTORIZATION

Factorization: If a polynomial p(x) can be expressed as p(x) = g(x) h (x), then each of the polynomials g(x) and h(x) is called a factor of p(x). The process of finding the factors is called factorization.

(a) Factorization of the Expression of the type ka + kb + kc.

Example

Factorize 5a-5b+5c

Solution

$5a - 5b + 5c = 5(a - b + c)$

Example

Factorize $5a - 5b - 15c$

Solution:

$5a - 5b - 15c = 5(a - b - 3c)$

(b) Factorization of the Expression of the type ac + ad + bc + bd

We can write ac + ad + bc + bd as

$(ac + ad) + (bc + db)$

$=a(c + d) + b(c + d)$

$=(a + b)(c + d)$

Example

Factorize $3x - 3a + xy - ay$

Solution:

Regrouping the terms of given polynomial

$3x + xy - 3a - ay = x(3 + y) - a(3 + y)$

$= (3 + y)(x - a)$

Example

Factorize $pqr + qr^2 - pr^2 - r^3$

Solution:

The given expression $= r(pq+qr-pr-r^2)$

$= r\left[(pq+qr) - pr - r^2\right]$

$= r\left[q(p+r) - r(p+r)\right]$

$= r(p+r)(q-r)$

(c) Factorization of the Expression of the type $a^2 \pm 2ab + b^2$.

We know that

(i) $a^2 + 2ab + b^2 = (a+b)^2 = (a+b)(a+b)$

(ii) $a^2 - 2ab + b^2 = (a-b)^2 = (a-b)(a-b)$

Example

Factorization $25x^2+16+40x$.

Solution:

$25x^2+40x+16 = (5x)^2+2(5x)(4) + (4)^2$

$= (5x+4)^2$

$= (5x+4)(5x+4)$

Example

Factorize $12x^2-36x+27$

Solution:

$12x^2 - 36x + 27 = 3(4x^2 - 12x + 9)$

$= 3\left[(2x)^2 - 2(2x)(3) + (3)^2\right]$

$= 3(2x-3)^2$

$= 3(2x-3)(2x-3)$

(d) Factorization of the Expression of the type $a^2 - b^2$.

Example Factorize

(i) $4x^2 - (2y-z)^2$ (ii) $6x^4 - 96$

Solution

(i) $4x^2-(2y-z)^2 = (2x)^2-(2y-z)^2$

$= [2x-(2y-z)][2x+(2y-z)]$

$= (2x-2y+z)(2x+2y-z)$

(ii) $6x^4-96 = 6(x^4-16)$

$= 6\left[(x^2)^2-(4)^2\right]$

$= 6(x^2-4)(x^2+4)$

$= 6\left[(x)^2-(2)^2\right](x^2+4)$

$= 6(x-2)(x+2)(x^2+4)$

(e) Factorization of the Expression of the types $a^2 \pm 2ab + b^2 - c^2$.

We know that

$$a^2 \pm 2ab + b^2 - c^2 = (a \pm b)^2 - (c)^2 = (a \pm b - c)(a \pm b + c)$$

Example

Factorize (i) $x^2+6x+9-4y^2$

(ii) $1+2ab-a^2-b^2$

Solution:

(i) $x^2+6x+9-4y^2 = (x+3)^2-(2y)^2$

$= (x+3+2y)(x+3-2y)$

(ii) $1+2ab-a^2-b^2 = 1-(a^2-2ab+b^2)$

$= (1)^2-(a-b)^2$

$= [1-(a-b)][1+(a-b)]$

$= (1-a+b)(1+a-b)$

Exercise 5.1

Q.1 Factorize

(i) $2abc-4abx+2abd$

$= 2ab(c-2x+d)$

(ii) $9xy-12x^2y+18y^2$

$= 3y(3x-4x^2+6y)$

(iii) $-3x^2y-3x+9xy^2$

$= -3x(xy+1-3y^2)$

(iv) $5ab^2c^3 - 10a^2b^3c + 20a^3bc^2$

$= 5abc(bc^2 - 2ab^2 + 4a^2c)$

(v) $3x^3y(x-3y) - 7x^2y^2(x-3y)$

$(x-3y)(3x^3y - 7x^2y^2)$

$(x-3y).x^2y(3x-7y)$

$\Rightarrow$ $x^2y(x-3y)(3x-7y)$

(vi) $2xy^3(x^2+5) + 8xy^2(x^2+5)$

$(x^2+5)(2xy^3 + 8xy^2)$

$(x^2+5)\ 2xy^2(y+4)$

$= 2xy^2(x^2+5)(y+4)$

Q.2 (i) $5ax - 3ay - 5bx + 3by$

$= 5ax - 5bx - 3ay + 3by$

$= 5x(a-b) - 3y(a-b)$

$= (a-b)(5x-3y)$

(ii) $3xy + 2y - 12x - 8$

$= 3xy - 12x + 2y - 8$

$= 3x(y-4) + 2(y-4)$

$= (y-4)(3x+2)$

(iii) $x^3 + 3xy^2 - 2x^2y - 6y^3$

$= x^3 - 2x^2y + 3xy^2 - 6y^3$

$= x^2(x-2y) + 3y^2(x-2y)$

$= (x-2y)(x^2+3y^2)$

(iv) $(x^2-y^2)z + (y^2-z^2)x$

$= x^2z - y^2z + y^2x - z^2x$

$= x^2z - z^2x + y^2x - y^2z$

$= xz(x-z) + y^2(x-z)$

$= (x-z)(xz+y^2)$

Q.3 (i) $144a^2 + 24a + 1$

$= (12a)^2 + 2(12a)(1) + (1)^2$

$= (12a+1)^2$

$= (12a+1)(12a+1)$

(ii) $\dfrac{a^2}{b^2} - 2 + \dfrac{b^2}{a^2}$

$= \left(\dfrac{a}{b}\right)^2 - 2\left(\dfrac{a}{b}\right)\left(\dfrac{b}{a}\right) + \left(\dfrac{b}{a}\right)^2$

$= \left(\dfrac{a}{b} - \dfrac{b}{a}\right)^2$

$= \left(\dfrac{a}{b} - \dfrac{b}{a}\right)\left(\dfrac{a}{b} - \dfrac{b}{a}\right)$

(iii) $(x+y)^2 - 14z(x+y) + 49z^2$

$= (x+y)^2 - 2(x+y)(7z) + (7z)^2$

$= (x+y-7z)^2$

$= (x+y-7z)(x+y-7z)$

(iv) $12x^2 - 36x + 27$

$= 3(4x^2 - 12x + 9)$

$= 3\left[(2x)^2 - 2(2x)(3) + (3)^2\right]$

$= 3(2x-3)^2$

$= 3(2x-3)(2x-3)$

Q.4 (i) $3x^2 - 75y^2$

$= 3(x^2 - 25y^2)$

$= 3\left[(x)^2 - (5y)^2\right]$

$= 3(x+5y)(x-5y)$

(ii) $x(x-1) - y(y-1)$

$= x^2 - x - y^2 + y$

$= x^2 - y^2 - x + y$

$= (x+y)(x-y) - 1(x-y)$

$= (x-y)(x+y-1)$

(iii) $128am^2 - 242an^2$

$= 2a(64m^2 - 121n^2)$

$= 2a\left[(8m)^2 - (11n)^2\right]$

$= 2a(8m + 11n)(8m - 11n)$

(iv) $3x - 243x^3$

$= 3x(1 - 81x^2)$

$= 3x\left[(1)^2 - (9x)^2\right]$

$= 3x(1 + 9x)(1 - 9x)$

Q.5 (i) $x^2 - y^2 - 6y - 9$

$= x^2 - (y^2 + 6y + 9)$

$= x^2 - \left[(y)^2 + 2(y)(3) + (3)^2\right]$

$= (x)^2 - (y + 3)^2$

$= [(x) + (y + 3)][(x) - (y + 3)]$

$= (x + y + 3)(x - y - 3)$

(ii) $x^2 - a^2 + 2a - 1$

$= x^2 - (a^2 - 2a + 1)$

$= (x)^2 - (a - 1)^2$

$= [(x) + (a - 1)][(x) - (a - 1)]$

$= (x + a - 1)(x - a + 1)$

(iii) $4x^2 - y^2 - 2y - 1$

$= 4x^2 - (y^2 + 2y + 1)$

$= (2x)^2 - (y + 1)^2$

$= [(2x) + (y + 1)][(2x - (y + 1)]$

$= (2x + y + 1)(2x - y - 1)$

(iv) $x^2 - y^2 - 4x - 2y + 3$

$= x^2 - y^2 - 4x - 2y + 4 - 1$

$= x^2 - 4x + 4 - y^2 - 2y - 1$

$= (x)^2 - 2(x)(2) + (2)^2 - (y^2 + 2y + 1)$

$= (x - 2)^2 - (y + 1)^2$

$= [(x - 2) + (y + 1)][(x - 2) - (y + 1)]$

$= (x - 2 + y + 1)(x - 2 - y - 1)$

$= (x + y - 1)(x - y - 3)$

(v) $25x^2 - 10x + 1 - 36z^2$

$= (5x)^2 - 2(5x)(1) + (1)^2 - (6z)^2$

$= (5x - 1)^2 - (6z)^2$

$= [(5x - 1) + (6z)][(5x - 1) - (6z)]$

$= (5x - 1 + 6z)(5x - 1 - 6z)$

$= (5x + 6z - 1)(5x - 6z - 1)$

(vi) $x^2 - y^2 - 4xz + 4z^2$

$= x^2 - 4xz + 4z^2 - y^2$

$= (x)^2 - 2(x)(2z) + (2z)^2 - (y)^2$

$= (x - 2z)^2 - (y)^2$

$= [(x - 2z) + (y)][(x - 2z) - (y)]$

$= (x - 2z + y)(x - 2z - y)$

(a) Factorization of the Expression of types $a^4 + a^2b^2 + b^4$ or $a^4 + 4b^4$

Factorization of such types of expression is explained in the following examples.

Example

Factorize $81x^4 + 36x^2y^2 + 16y^4$

Solution

$81x^4 + 36x^2y^2 + 16y^4$

$= (9x^2)^2 + 72x^2y^2 + (4y^2)^2 - 36x^2y^2$

$= (9x^2)^2 + (4y^2)^2 + 2(9x^2)(4y^2) - 36x^2y^2$

$=(9x^2+4y^2)^2-(6xy)^2$

$=(9x^2+4y^2+6xy)(9x^2+4y^2-6xy)$

$=(9x^2+6xy+4y^2)(9x^2-6xy+4y^2)$

Example

Factorize $9x^4+36y^4$

Solution:

$9x^4+36y^4$

$=9x^4+36y^4+36x^2y^2-36x^2y^2$

$=(3x^2)^2+2(3x^2)(6y^2)+(6y^2)^2-(6xy)^2$

$=(3x^2+6y^2)^2-(6xy)^2$

$=(3x^2+6y^2+6xy)(3x^2+6y^2-6xy)$

$=(3x^2+6xy+6y^2)(3x^2-6xy+6y^2)$

(b) Factorization of the Expression of the type x^2+px+q.

Example

Factorize (i) $x^2-7x+12$

(ii) $x^2+15x-36$

Solution:

(i) $x^2-7x+12$

From the factors of 12 the suitable pair of numbers is –3 and –4 since

$(-3)+(-4)=-7$ *and* $(-3)(-4)=12$

Hence $x^2-7x+12=x^2-3x-4x+12$

$=x(x-3)-4(x-3)$

$=(x-3)(x-4)$

(ii) $x^2+5x-36$

From the possible factors of 36, the suitable pair is 9 and –4 because $9+(-4)=5$ and $9\times(-4)=-36$

Hence $x^2+5x-36 \quad =x^2+9x-4x-36$

$=x(x+9)-4(x+9)$

$=(x+9)(x-4)$

(c) Factorization of the Expression of the type $ax^2+bx+c,\ a\neq 0$

Example

Factorize (i) $9x^2+21x-8$

(ii) $2x^2-8x-42$

(iii) $10x^2-41xy+21y^2$

Solution:

(i) $9x^2+21x-8$

In this case, on comparing with $ax^2+bx+c,\ ac=(9)(-8)=-72$.

From the possible factors of 72 the suitable pair of numbers (with proper sign) is 24 and –3 whose Sum = 24 + (–3) = 21, (the coefficient of x)

And their product = (24) (–3) = –72 = ac

Hence $9x^2+21x-8$

$=9x^2+24x-3x-8$

$=3x(3x+8)-1(3x+8)$

$=(3x+8)(3x-1)$

(ii) $2x^2-8x-42=2(x^2-4x-21)$

Comparing

$x^2-4x-21$ *with* ax^2+bx+c

We have ac = (+1) (–21) = –21

From the possible factors of 21 the suitable pair of numbers is –7 and +3 whose

Sum =–7+3=–4 and product =(–7)(3)=–21

Hence $x^2-4x-21$

$=x^2+3x-7x-21$

$=x(x+3)-7(x+3)$

$= (x+3)(x-7)$

Hence $2x^2 - 8x - 42 = 2(x^2 - 4x - 21)$
$= 2(x+3)(x-7)$

(iii) $10x^2 - 41xy + 21y^2$

Here $ac = (10)(21) = 210$

Two suitable factors of 210 are –35 and –6.

Their sum = –35 –6 = -41

And product = (–35) (–6) = 210

Hence $10x^2 - 41xy + 21y^2$

$$= 10x^2 - 35xy - 6xy + 21y^2$$
$$= 5x(2x-7y) - 3y(2x-7y)$$
$$= (2x-7y)(5x-3y)$$

(d) Factorization of the following types of Expressions.

$(ax^2 + b + c)(ax^2 + bx + d) + k$

$(x+a)(x+b)(x+c)(x+d) + k$

$(x+a)(x+b)(x+c)(x+d) + kx^2$

Example

Factorize $(x^2 - 4x - 5)(x^2 - 4x - 12) - 144$

Solution:

$(x^2 - 4x - 5)(x^2 - 4x - 12) - 144$

Let $y = x^2 - 4x$. Then

$$\begin{aligned}
(y-5)(y-12) - 144 &= y^2 - 17y + 60 - 144 \\
&= y^2 - 17y - 84 \\
&= y^2 - 21y + 4y - 84 \\
&= y(y-21) + 4(y-21) \\
&= (y-21)(y+4) \\
&= (x^2 - 4x - 21)(x^2 - 4x + 4) \quad (\textit{Since } y = x^2 - 4x) \\
&= (x^2 - 7x + 3x - 21)\left[(x)^2 - 2(x)(2) + (2)^2\right] \\
&= \left[x(x-7) + 3(x-7)\right](x-2)^2 \\
&= (x-7)(x+3)(x-2)(x-2)
\end{aligned}$$

Example

Factorize

$(x+1)(x+2)(x+3)(x+4)-120$

Solution:

We observe that $1+4=2+3$.

It suggests that we rewrite the given expression as

$[(x+1)(x+4)][(x+2)(x+3)]-120$

$(x^2+5x+4)(x^2+5x+6)-120$

Let $x^2+5x=y$, then

We get $(y+4)(y+6)-120$

$=y^2+10y+24-120$

$=y^2+10y-96$

$=y^2+16y-6y-96$

$=y(y+16)-6(y+16)$

$=(y+16)(y-6)$

$=(x^2+5x+16)(x^2+5x-6)$ (since $y=x^2+5x$)

$=(x^2+5x+16)\left[x^2+6x-x-6\right]$

$=(x^2+5x+16)[(x+6)-1(x+6)]$

$=(x^2+5x+16)(x+6)(x-1)$

Example

Factorize $(x^2-5x+6)(x^2+5x+6)-2x^2$

Solution:

$(x^2-5x+6)(x^2+5x+6)-2x^2$

$=\left[x^2-3x-2x+6\right]\left[x^2+3x+2x+6\right]-2x^2$

$=[x(x-3)-2(x-3)]\,[x(x+3)+2(x+3)]-2x^2$

$=[(x-3)(x-2)][(x+3)(x+2]-2x^2$

$=[(x-2)(x+2)][(x-3)(x+3)]-2x^2$

$=(x^2-4)(x^2-9)-2x^2$

$=x^4-13x^2+36-2x^2$

$=x^4-15x^2+36$

$=x^4-12x^2-3x^2+36$

$=x^2(x^2-12)-3(x^2-12)$

$=(x^2-12)(x^2-3)$

$=\left[(x)^2-(2\sqrt{3})^2\right]\left[(x)^2-(\sqrt{3})^2\right]$

$=(x-2\sqrt{3})(x+2\sqrt{3})(x-\sqrt{3})(x+\sqrt{3})$

(e) Factorization of Expressions of the following Types

$a^3+3a^2b+3ab^2+b^3$

$a^3-3a^2b+3ab^2-b^3$

Example:

Factorize $x^3-8y^3-6x^2y+12xy^2$

Solution:

$x^3-8y^3-6x^2y+12xy^2$

$=(x)^3-(2y)^3-3(x)^2(2y)+3(x)(2y)^2$

$=(x)^3-3(x)^2(2y)+3(x)(2y)^2-(2y)^3$

$=(x-2y)^3$

$=(x-2y)(x-2y)(x-2y)$

(d) Factorization of Expressions of the following types $a^3 \pm b^3$

We recall the formulas,

$a^3+b^3=(a+b)(a^2-ab+b^2)$

$a^3-b^3=(a-b)(a^2+ab+b^2)$

Example

Factorize $27x^3+64y^3$

Solution:

$27x^3+64y^3 \quad =(3x)^3+(4y)^3$

$=(3x+4y)\left[(3x)^2-(3x)(4y)+(4y)^2\right]$

$=(3x+4y)(9x^2-12xy+16y^2)$

Example

Factorize $1-125x^3$

Solution

$$1-25x^3 \quad =(1)^3-(5x)^3$$

$$=(1-5x)\left[(1)^2+(1)(5x)+(5x)^2\right]$$

$$=(1-5x)(1+5x+25x^2)$$

Exercise 5.2

Q.1 Factorize

(i) $x^4+\dfrac{1}{x^4}-3$

$$=x^4+\frac{1}{x^4}-2-1$$

$$=(x^2)^2+\left(\frac{1}{x^2}\right)^2-2(x^2)\left(\frac{1}{x^2}\right)-1$$

$$=\left(x^2-\frac{1}{x^2}\right)^2-(1)^2$$

$$=\left(x^2-\frac{1}{x^2}+1\right)\left(x^2-\frac{1}{x^2}-1\right)$$

(ii) $3x^4+12y^4$

$$=3(x^4+4y^4)$$

$$=3\left[(x^2)^2+(2y^2)^2+2(x^2)(2y^2)-4x^2y^2\right]$$

$$=3\left[(x^2+2y^2)^2-(2xy)^2\right]$$

$$=3(x^2+2y^2+2xy)(x^2+2y^2-2xy)$$

$$=3(x^2+2xy+2y^2)(x^2-2xy+2y^2)$$

(iii) $a^4+3a^2b^2+4b^4$

$$a^4+4a^2b^2+4b^4-a^2b^2$$

$$=(a^2)^2+2(a^2)(2b^2)+(2b^2)^2-a^2b^2$$

$$=(a^2+2b^2)^2-(ab)^2$$

$$=(a^2+2b^2+ab)(a^2+2b^2-ab)$$

$$=(a^2+ab+2b^2)(a^2-ab+2b^2)$$

(iv) $4x^4+81$

$$=(2x^2)^2+(9)^2+2(2x^2)(9)-36x^2$$

$$=(2x^2+9)^2-(6x)^2$$

$$=(2x^2+9+6x)(2x^2+9-6x)$$

$$=(2x^2+6x+9)(2x^2-6x+9)$$

(v) x^4+x^2+25

$$=(x^2)^2+2(x^2)(5)+(5)^2-9x^2$$

$$=(x^2+5)^2-(3x)^2$$

$$=(x^2+5+3x)(x^2+5-3x)$$

$$=(x^2+3x+5)(x^2-3x+5)$$

(vi) x^4+4x^2+16

$$=(x^2)^2+2(x^2)(4)+(4)^2-4x^2$$

$$=(x^2+4)^2-(2x)^2$$

$$=(x^2+4+2x)(x^2+4-2x)$$

$$=(x^2+2x+4)(x^2-2x+4)$$

Q.2 (i) $x^2+14x+48$

$$=x^2+6x+8x+48$$

$$=x(x+6)+8(x+6)$$

$$=(x+6)(x+8)$$

(ii) $x^2-21x+108$

$$=x^2-9x-12x+108$$

$$=x(x-9)-12(x-9)$$

$=(x-9)(x-12)$

(iii) $x^2-11x-42$

$=x^2+3x-14x-42$

$=x(x+3)-14(x+3)$

$=(x+3)(x-14)$

(iv) $x^2+x-132$

$=x^2+12x-11x-132$

$=x(x+12)-11(x+12)$

$=(x+12)(x-11)$

Q.3 (i) $4x^2+12x+5$

$=4x^2+2x+10x+5$

$=2x(2x+1)+5(2x+1)$

$=(2x+1)(2x+5)$

(ii) $30x^2+7x-15$

$=30x^2+25x-18x-15$

$=5x(6x+5)-3(6x+5)$

$=(6x+5)(5x-3)$

(iii) $24x^2-65x+21$

$=24x^2-56x-9x+21$

$=8x(3x-7)-3(3x-7)$

$=(3x-7)(8x-3)$

(iv) $5x^2-16x-21$

$=5x^2+5x-21x-21$

$=5x(x+1)-21(x+1)$

$=(x+1)(5x-21)$

(v) $4x^2-17xy+4y^2$

$=4x^2-16xy-xy+4y^2$

$=4x(x-4y)-y(x-4y)$

$=(x-4y)(4x-y)$

(vi) $3x^2-38xy-13y^2$

$=3x^2-39xy+xy-13y^2$

$=3x(x-13y)+y(x-13y)$

$=(x-13y)(3x+y)$

(vii) $5x^2+33xy-14y^2$

$=5x^2+35xy-2xy-14y^2$

$=5x(x+7y)-2y(x+7y)$

$=(x+7y)(5x-2y)$

(viii) $\left(5x-\frac{1}{x}\right)^2+4\left(5x-\frac{1}{x}\right)+4$

$=\left(5x-\frac{1}{x}\right)^2+2\left(5x-\frac{1}{x}\right)(2)+(2)^2$

$=\left(5x-\frac{1}{x}+2\right)^2$

$=\left(5x-\frac{1}{x}+2\right)\left(5x-\frac{1}{x}+2\right)$

Q.4 (i) $(x^2+5x+4)(x^2+5x+6)-3$

Let $x^2+5x=y$

then

$(x^2+5x+4)(x^2+5x+6)-3$

$=(y+4)(y+6)-3$

$=y^2+4y+6y+24-3$

$=y^2+10y+21$

$=y^2+3y+7y+21$

$=y(y+3)+7(y+3)$

$=(y+3)(y+7)$

Putting value of y

$=(x^2+5x+3)(x^2+5x+7)$

(ii) $(x^2-4x)(x^2-4x-1)-20$

Let $x^2-4x=y$

then

$$(x^2-4x)(x^2-4x-1)-20$$
$$=y(y-1)-20$$
$$=y^2-y-20$$
$$=y^2+4y-5y-20$$
$$=y(y+4)-5(y+4)$$
$$=(y+4)(y-5)$$

Putting value of y

$$=(x^2-4x+4)(x^2-4x-5)$$
$$=\left[(x)^2-2(x)(2)+(2)^2\right]\left[x^2+x-5x-5\right]$$
$$=(x-2)^2\left[x(x+1)-5(x+1)\right]$$
$$=(x-2)^2(x+1)(x-5)$$

(iii) $(x+2)(x+3)(x+4)(x+5)-15$

$$=\left[(x+2)(x+5)\right]\left[(x+3)(x+4)\right]-15$$
$$=(x^2+2x+5x+10)(x^2+3x+4x+12)-15$$
$$=(x^2+7x+10)(x^2+7x+12)-15$$

Let $x^2+7x=y$

$$=(y+10)(y+12)-15$$
$$=y^2+10y+12y+120-15$$
$$=y^2+22y+105$$
$$=y^2+7y+15y+105$$
$$=y(y+7)+15(y+7)$$
$$=(y+7)(y+15)$$

Putting value of 'y'

$$(x^2+7x+7)(x^2+7x+15)$$

(iv) $(x+4)(x-5)(x+6)(x-7)-504$

$$=(x^2+4x-5x-20)(x^2+6x-7x-42)-504$$
$$=(x^2-x-20)(x^2-x-42)-504$$

Let $x^2-x=y$

$$=(y-20)(y-42)-504$$
$$=y^2-20y-42y+840-504$$
$$=y^2-62y+336$$
$$=y^2-6y-56y+336$$
$$=y(y-6)-56(y-6)$$
$$=(y-6)(y-56)$$

Putting value of 'y'

$$=(x^2-x-6)(x^2-x-56)$$
$$=(x^2+2x-3x-6)(x^2+7x-8x-56)$$
$$=\left[x(x+2)-3(x+2)\right]\left[x(x+7)-8(x+7)\right]$$
$$=(x+2)(x-3)(x+7)(x-8)$$

(v) $(x+1)(x+2)(x+3)(x+6)-3x^2$

$$=(x+1)(x+6)(x+2)(x+3)-3x^2$$
$$=(x^2+x+6x+6)(x^2+2x+3x+6)-3x^2$$
$$=(x^2+6+7x)(x^2+6+5x)-3x^2$$
$$=\frac{x^2}{x^2}\left[(x^2+6+7x)(x^2+6+5x)-3x^2\right]$$
$$=x^2\left[\frac{(x^2+6+7x)(x^2+6+5x)}{x^2}-\frac{3x^2}{x^2}\right]$$
$$=x^2\left[\left(x+\frac{6}{x}+7\right)\left(x+\frac{6}{x}+5\right)-3\right]$$

Let $x+\frac{6}{x}=y$

$$=x^2\left[(y+7)(y+5)-3\right]$$
$$=x^2(y^2+7y+5y+35-3)$$
$$=x^2(y^2+12y+32)$$
$$=x^2(y^2+4y+8y+32)$$
$$=x^2\left[y(y+4)+8(y+4)\right]$$
$$=x^2(y+4)(y+8)$$

Putting value of y

$$=x^2\left(x+\frac{6}{x}+4\right)\left(x+\frac{6}{x}+8\right)$$

$$=x^2\left(\frac{x^2+4x+6}{x}\right)\left(\frac{x^2+8x+6}{x}\right)$$

$$=(x^2+4x+6)(x^2+8x+6)$$

$$=(x^2+4x+6)(x^2+8x+6)$$

Q.5

(i) $x^3+48x-12x^2-64$

$$=x^3-12x^2+48x-64$$
$$=(x)^3-3(x^2)(4)+3(x)(4)^2-(4)^3$$
$$=(x-4)^3$$
$$=(x-4)(x-4)(x-4)$$

(ii) $8x^3+60x^2+150x+125$

$$=(2x)^3+3(2x)^2(5)+3(2x)(5)^2+(5)^3$$
$$=(2x+5)^3$$
$$=(2x+5)(2x+5)(2x+5)$$

(iii) $x^3-18x^2+108x-216$

$$=(x)^3-3(x)^2(6)+3(x)(6)^2-(6)^3$$
$$=(x-6)^3$$
$$=(x-6)(x-6)(x-6)$$

(iv) $8x^3-125y^3-60x^2y+150xy^2$

$$=8x^3-60x^2y+150xy^2-125y^3$$
$$=(2x)^3-3(2x)^2(5y)+3(2x)(5y)^2-(5y)^3$$
$$=(2x-5y)^3$$
$$=(2x-5y)(2x-5y)(2x-5y)$$

Q.6 (i) $27+8x^3$

$$=(3)^3+(2x)^3$$
$$=(3+2x)\left[(3)^2-(3)(2x)+(2x)^2\right]$$
$$=(3+2x)(9-6x+4x^2)$$

or $=(2x+3)(4x^2-6x+9)$

(ii) $125x^3-216y^3$

$$=(5x)^3-(6y)^3$$
$$=(5x-6y)\left[(5x)^2+(5x)(6y)+(6y)^2\right]$$
$$=(5x-6y)(25x^2+30xy+36y^2)$$

(iii) $64x^3+27y^3$

$$=(4x)^3+(3y)^3$$
$$=(4x+3y)\left[(4x)^2-(4x)(3y)+(3y)^2\right]$$
$$=(4x+3y)(16x^2-12xy+9y^2)$$

(iv) $8x^3+125y^3$

$$=(2x)^3+(5y)^3$$
$$=(2x+5y)\left[(2x)^2-(2x)(5y)+(5y)^2\right]$$
$$=(2x+5y)(4x^2-10xy+25y^2)$$

Remainder Theorem

If a polynomial p(x) is divided by a linear divisor $(x-a)$, then the remainder is $p(a)$.

Proof

Let $q(x)$ be the quotient obtained after dividing $p(x)$ by $(x-a)$. But the divisor $(x-a)$ is linear. So the remainder must be of degree zero i.e., a non-zero constant, say R. Consequently, by division Algorithm we may write.

$$p(x)=(x-a)q(x)+R$$

This is an identity in x and so is true for all real numbers x. In particular, it is true for $x=a$. Therefore,

$$p(a)=(a-a)q(a)+R=0+R=R$$

i.e., $p(a)=$ the remainder.

Hence the theorem.

Note: Similarly, if the divisor is (ax–b), we have

$$p(x)=(ax-b)q(x)+R$$

Substituting $x = \frac{b}{a}$ so that ax − b = 0, we obtain

$$p\left(\frac{b}{a}\right) = 0.\ q\left(\frac{b}{a}\right) + R = 0 + R = R$$

Thus if the divisor is linear, the above theorem provides an efficient way of finding the remainder without being involved in the process of long division.

To find remainder (without dividing) when a polynomial is divided by a Linear Polynomial

Example

Find the remainder when $9x^2 - 6x + 2$ is divided by

(i) $x-3$ (ii) $x+3$

(iii) $3x+1$ (iv) x

Solution:

Let $p(x) = 9x^2 - 6x + 2$

(i) When $p(x)$ is divided by $x-3$, by Remainder Theorem, the remainder is:

$$R = p(3) = 9(3)^2 - 6(3) + 2 = 65$$
$$= 9(9) - 18 + 2$$
$$P(3) = 81 - 16$$
$$= 65$$

(ii) When $p(x)$ is divided by $x+3 = x-(-3)$, the remainder is

$$R = p(-3) = 9(-3)^2 - 6(-3) + 2$$
$$= 9(9) + 18 + 2$$
$$= 81 + 20 = 101$$

(iii) When $p(x)$ is divided by $3x+1$, the remainder is

$$R = p\left(-\frac{1}{3}\right) = 9\left(-\frac{1}{3}\right)^2 - 6\left(-\frac{1}{3}\right) + 2 = 5$$

(iv) When $p(x)$ is divided by x, the remainder is

$$R = p(0) = 9(0)^2 - 6(0) + 2 = 2$$

Example

Find the value of k is the expression $x^3 + kx^2 + 3x - 4$ leaves a remainder of -2 when divided by $x+2$.

Solution:

Let $p(x) = x^3 + kx^2 + 3x - 4$.

By the remainder Theorem, when $p(x)$ is divided by $x+2 = x-(-2)$, the remainder is:

$$p(-2) = (-2)^3 + k(-2)^2 + 3(-2) - 4$$
$$= -8 + 4k - 6 - 4$$
$$= 4k - 18$$

By the given condition, we have

$p(-2) = -2 \quad \Rightarrow \quad 4k - 18 = -2$

$\Rightarrow \quad k = 4$

5.2.3 Zero of a polynomial

If a specific number $x = a$ is substituted for a variable x in a polynomial $p(x)$ so that the value $p(a)$ is zero, then $x = a$ is called a zero of the polynomial $p(x)$.

Factor Theorem

The polynomial $(x-a)$ is a factor of the polynomial $p(x)$ if and only if $p(a) = 0$.

Proof:

Let $q(x)$ be the quotient and R the remainder when a polynomial $p(x)$ is divided by $(x-a)$. Then by division Algorithm,

$$p(x) = (x-a)q(x) + R$$

By the Remainder Theorem, R = p(a).

Hence $p(x)=(x-a)q(x)+p(a)$

(i) Now if $p(a)=0$, then $p(x)=(x-a)q(x)$

i.e., $(x-a)$ is a factor of $p(x)$.

(ii) Conversely, if $(x-a)$ is a factor of $p(x)$, then the remainder upon dividing $p(x)$ by $(x-a)$ must be zero i.e., p(a) = 0.

Example

Determine if $(x-2)$ is a factor of x^3-4x^2+3x+2.

Solution:

Let

$p(x)=x^3-4x^2+3x+2$

Then the remainder for $(x-2)$ is:

$p(2)=(2)^3-4(2)^2+3(2)+2$

$=8-16+6+2=0$

Hence by Factor Theorem, $(x-2)$ is a factor of the polynomial $p(x)$.

Example

Find a polynomial $p(x)$ of degree 3 that has 2, −1, and 3 as zeros (i.e., roots).

Solution:

Since $x=2,-1,3$ are roots of $p(x)=0$.

So by Factor theorem $(x-2),(x+1)$ *and* $(x-3)$ are the factors of $p(x)$.

Thus $p(x)=a(x-2)(x+1)(x-3)$

Where any non-zero value can be assigned to a.

Taking a = 1, we get

$p(x)=(x-2)(x+1)(x-3)$

$=x^3-4x^2+x+6$ as the required polynomial.

Exercise 5.3

Q.1 Use the remainder theorem to find the remainder, when.

(i) $3x^3-10x^2+13x-6$ is divided by $(x-2)$

Sol:

Let $P(x)=3x^3-10x^2+13x-6$

When P(x) is divided by x – 2 by remainder theorem, the remainder is:

$R=P(2)=3(2)^3-10(2)^2+13(2)-6$

$=3(8)-10(4)+26-6$

$=24-40+26-6$

$=50-46$

$=4$

(ii) $4x^3-4x+3$ is divided by $(2x-1)$

Sol:

Let $P(x)=4x^3-4x+3$ when P(x) is divided by 2x – 1 by remainder theorem, the remainder is

$R=P\left(\frac{1}{2}\right)$ $=4\left(\frac{1}{2}\right)^3-4\left(\frac{1}{2}\right)+3$

$=4\left(\frac{1}{8}\right)-2+3$

$=\frac{1}{2}+1$

$=\frac{1+2}{2}$

$R \quad = \frac{3}{2}$

(iii) $6x^4+2x^3-x+2$ is divided by $(x + 2)$

Sol:

Let P (x) = $6x^4+2x^3-x+2$ when P(x) is divided by x + 2 by remainder theorem, the remainder is

$$R=P(-2) \quad =6(-2)^4+2(-2)^3-(-2)+2$$
$$=6(16)+2(-8)+2+2$$
$$=96-16+4$$
$$=80+4$$
$$R \quad =84$$

(iv) $(2x-1)^3+6(3+4x)^2-10$ is divided by 2x + 1

Sol:

Let p(x) = $(2x-1)^3+6(3+4x)^2-10$ when P(x) is divided by 2x + 1 by remainder theorem, then remainder is

$$R=p\left(-\frac{1}{2}\right)=\left[2\left(-\frac{1}{2}\right)-1\right]^3+6\left[3+4\left(-\frac{1}{2}\right)\right]^2-10$$
$$=(-1-1)^3+6(3-2)^2-10$$
$$=(-2)^3+6(1)^2-10$$
$$=-8+6-10$$
$$=-12$$

(v) $x^3-3x^2+4x-14$ is divided by x + 2

Sol:

Let P(x) = $x^3-3x^2+4x-14$ when P(x) is divided by x + 2 by remainder theorem, then remainder is

$$R=P(-2) \quad =(-2)^3-3(-2)^2+4(-2)-14$$
$$=-8-3(4)-8-14$$
$$=-8-12-8-14$$
$$=-42$$

Q.2.

(i) If (x+2) is a factor of $3x^2-4kx-4k^2$, then find the value(s) of k.

Sol:

Let $P(x)=3x^2-4kx-4k^2$

As given that x + 2 is a factor of P(x), so

$R=0$

i.e. $P(-2)=0$

So $3(-2)^2-4k(-2)-4k^2=0$

$12+8k-4k^2=0$

Dividing by 4

$3+2k-k^2=0$

$3+3k-k-k^2=0$

$3(1+k)-k(1+k)=0$

$(1+k)(3-k)=0$

$\Rightarrow 1+k=0 \text{ or } 3-k=0$

$\Rightarrow k=-1 \text{ or } k=3$

(ii) If (x −1) is factor of $x^3-kx^2+11x-6$ then find the value of k.

Sol:

$P(x)=x^3-kx^2+11x-6$

As given that x − 1 is a factor of P(x), so

$R \quad = \quad 0$

$P(1) \quad = \quad 0$

$(1)^3-k(1)^2+11(1)-6 = 0$

$1-k+11-6 = 0$

$6-k = 0$

$\Rightarrow k = 6$

Q.3 Without actual long division determine whether

(i) (x − 2) and (x − 3) are factors of P

$P(x)=x^3-12x^2+44x-48$

Sol:

$P(x)=x^3-12x^2+44x-48$

Taking $x-2$

$R=P(2)$

$=(2)^3-12(2)^2+44(2)-48$

$=8-12(4)+88-48$

$=8-48+88-48$

$=0$

As the remainder is zero, so $(x-2)$ is a factor of P(x)

Now $P(x)=x^3-12x^2+44x-48$

Taking $x-3$

$R=P(3)$

$=(3)^3-12(3)^2+44(3)-48$

$=27-12(9)+132-48$

$=27-108+132-148$

$=3\neq 0$

As the remainder is not equal to zero, so $(x-3)$ is not a factor of P(x).

(ii) $(x-2)$, $(x+3)$ and $(x-4)$ are factors of $q(x)=x^3+2x^2-5x-6$

Sol:

$q(x)=x^3+2x^2-5x-6$

Taking $x-2$

$R=q(2)=(2)^3+2(2)^2-5(2)-6$

$=8+2(4)-10-6$

$R=0$

As the remainder is zero so $(x-2)$ is a factor of P(x)

Now $q(x)=x^3+2x^2-5x-6$

Taking $x+3$

$R=q(-3)$

$=(-3)^3+2(-3)^2-5(-3)-6$

$=-27+2(9)+15-6$

$=-27+18+15-6$

$=0$

As the remainder is zero, so $(x+3)$ is a factor of P(x)

Now $q(x)=x^3+2x^2-5x-6$

Taking $x-4$

$R=q(4)$

$=(4)^3+2(4)^2-5(4)-6$

$=64+2(16)-20-6$

$=64+32-20-6$

$=70\neq 0$

As remainder is not equal to zero, so $x-4$ is not a factor of P(x)

Q.4 For what value of m is the polynomial $P(x)=4x^3-7x^2+6x-3m$ exactly divisible by x + 2?

Sol:

$m=?$

$P(x)=4x^3-7x^2+6x-3m$

Taking $x+2$

As $p(x)$ is exactly divisible by $(x+2)$, so

$R=0$

$P(-2)=0$

$4(-2)^3-7(-2)^2+6(-2)-3m=0$

$4(-8)-7(4)-12-3m=0$

$-32-28-12-3m=0$

$-72-3m=0$

$-3m=+72$

$m=\frac{72}{-3}$

$m=-24$

Q.5 Determine the value of k if $P(x)=kx^3+4x^2+3x-4$ and $q(x)=x^3-4x+k$. Leaves the same remainder when divided by x – 3.

Sol:

K = ?

When p(x) is divided by (x−3) by remainder theorem then remainder is

$R_1=P(3)$

$=k(3)^3+4(3)^2+3(3)-4$

$=27k+36+9-4$

$=27k+41$

When q(x) is divided by (x−3) by remainder theorem then remainder is

$R_2=q(3)$

$q(x)=x^3-4x+k$

$=(3)^3-4(3)+k$

$=27-12+k$

$=15+k$

As given that when P(x) and q(x) are divided by x – 3, then remainder is same, so

$R_1 = R_2$

$27k+41=15+k$

$27k-k=15-41$

$26k=-26$

$k=\frac{-26}{26}$

$\boxed{k=-1}$

Q.6

The remainder of dividing the polynomial $P(x)=x^3+ax^2+7$ by (x + 1) is 2b. calculate the value of 'a' and 'b' if this expression leaves a remainder of (b + 5) on being divided by (x – 2)

Sol:

$P(x)=x^3+ax^2+7$

The remainder by dividing

$P(x)$ by $x+1$ is $2b$, so

$P(-1)=2b$

$(-1)^3+a(-1)^2+7=2b$

$-1+a+7=2b$

$a+6=2b$

$a-2b=-6$......(i)

Taking $x-2$

The remainder by dividing

$P(x)$ by $(x-2)$ is $(b+5)$, so

$P(2)=b+5$

$(2)^3+a(2)^2+7=b+5$

$8+4a+7=b+5$

$4a+15=b+5$

$4a-b=5-15$

$4a-b=-10$ .....(ii)

Multiplying (ii) by 2

$8a-2b=-20$ ..... (iii)

By Subtracting , (iii) from (i)

$$\begin{array}{rrr} a & -2b & =-6 \\ 8a & \mp 2b & =\mp 20 \\ \hline -7a & & =14 \end{array}$$

$a=-\frac{14}{7}=-2$

Putting (1)

$a-2b=-6$
$-2-2b=-6$
$-2b=-6+2$
$-2b=-4$
$b=2$

Q.7 The polynomial $x^3+\ell x^2+mx+24$ has a factor (x + 4) and it leaves a remainder of 36 when divided by (x–2). Find the value of ℓ and m.

Sol:

Let $P(x)=x^3+\ell x^2+mx+24$

As $(x+4)$ is a factor of $P(x)$,

So remainder will be zero. i.e

$R=P(-4)=0$

$P(-4)=0$

$(-4)^3+\ell(-4)^2+m(-4)+24=0$

$-64+16\ell-4m+24=0$

$16\ell-4m-40=0$

$16\ell-4m=40$

Dividing by 4

$4\ell-m=10$.....(i)

Now as given that P(x) is divided by (x – 2) leaves a remainder 36, so

$R=36$

i.e. $P(2)=36$

$(2)^3+\ell(2)^2+m(2)+24=36$

$8+4\ell+2m+24=36$

$4\ell+2m+32=36$

$4\ell+2m=36-32$

$4\ell+2m=4$

Dividing by 2

$2\ell+m=2$.......(ii)

Adding (i) and (ii)

$$\begin{array}{l} 4\ell-m=10 \\ \underline{2\ell+m=2} \\ 6\ell \quad\ =12 \end{array}$$

$\ell \quad -\dfrac{12}{6}$

$\ell \quad =2$

Putting value of 'ℓ' in (ii)

$2\ell+m=2$

$2(2)+m=2$

$m=2-4$

$m=-2$

Q.8. The Expression ℓx^3+mx^2-4 leaves remainder of –3 and 12 when divided by (x–1) and (x+2) respectively. Calculate the values of ℓ and m.

Sol:

Let $P(x)=\ell x^3+mx^2-4$

As given that P(x) when divided by x – 1 leaves remainder –3, so

$R=-3$

$P(1)=-3$

$\ell(1)^3+m(1)^2-4=-3$

$\ell+m-4=-3$

$\ell+m=4-3$

$\ell+m=1$...........(i)

As given that P(x) when divided by (x + 2) leaves the remainder 12, so

$R=12$

$P(-2)=12$

$\ell(-2)^3+m(-2)^2-4=12$

$-8\ell+4m-4=12$

$-8\ell+4m=12+4$

$-8\ell+4m=16$

Dividing by 4

$-2\ell+m=4$........(ii)

Subtracting (ii) from (i)

$$\begin{array}{r} \ell+m=1 \\ -2\ell+m=4 \\ +\quad -\quad - \\ \hline 3\ell \quad =-3 \end{array}$$

$$\ell=\frac{-3}{3}$$

$$\ell=-1$$

Putting value of 'ℓ' in (i)

$\ell+m=1$

$-1+m=1$

$m=1+1$

$m=2$

Q.9 The expression $ax^3-9x^2+bx+3a$ is exactly divisible by x^2-5x+6. Find the values of a and b

Sol:

Let $P(x)=ax^3-9x^2+bx+3a$

Taking x^2-5x+6

$=x^2-2x-3x+6$

$=x(x-2)-3(x-2)$

$=(x-2)(x-3)$

As given that P(x) is exactly divisible by (x – 2), so $P(2)=0$

$a(2)^3-9(2)^2+b(2)+3a=0$

$8a-36+2b+3a=0$

$11a+2b=36$.........(i)

As given that P(x) is exactly divisible by x–3, so

$P(3)=0$

$a(3)^3-9(3)^2+b(3)+3a=0$

$27a-81+3b+3a=0$

$30a+3b=81$

Dividing by 3

$10a+b=27$......(ii)

Multiplying (ii) by 2 and subtracting (i) from it.

$$\begin{array}{r} 20a+2b=54 \\ 11a+2b=36 \\ -\quad -\quad - \\ \hline 9a \quad =18 \end{array}$$

$$a=\frac{18}{9}$$

$$a=2$$

Putting value of 'a' in (ii)

$10a+b=27$

$10(2)+b=27$

$b=27-20$

$b=7$

Rational Root Theorem

Let

$a_0x^n+a_1x^{n-1}+.......+a_{n-1}x+a_n=0, \quad a_0\neq 0$

be a polynomial equation of degree n with integral coefficients. If p/q is a rational root (expressed in lowest terms) of the equation, then p is a factor of the constant term a_n and q is a factor of the leading coefficient a_0.

Example

Factorize the polynomial x^3-4x^2+x+6, by using Factor Theorem.

Solution:

We have $P(x)=x^3-4x^2+x+6$.

Possible factors of the constant term p = 6 are ±1, ±2, ±3, and ±6 and of leading coefficient q = 1 are ±1. Thus the expected zeros (or roots) of $P(x)=0$ are $\frac{p}{q}=\pm1,\pm2,\pm3$ and ±6. If $x=a$ is a zero of $P(x)$, then $(x-a)$ will be a factor.

We use the hit and trial method to find zeros of P(x). Let us try x = 1.

Now $P(1)=(1)^3-4(1)^2+1+6$

$=1-4+1+6$

$=4\neq0$

Hence x = 1 is not a zero of P(x).

Again $P(-1)=(-1)^3-4(-1)^2-1+6$

$=-1-4-1+6=0$

Hence $x=-1$ is a zero of P(x) and therefore,

$x-(-1)=(x+1)$ is a factor of P(x).

Now $P(2)=(2)^3-4(2)^2+2+6$

$=8-16+2+6=0\Rightarrow x=2$ is a root.

Hence $(x-2)$ is also a factor of P(x).

Similarly $P(3)=(3)^3-4(3)^2+3+6$

$=27-36+3+6=0\Rightarrow x=3$ is a zero of P(x).

Hence $(x-3)$ is the third factor of P(x).

Thus the factorized form of

$P(x)=x^3-4x^2+x+6$ *is*

$(x+1)(x-2)(x-3)$.

Exercise 5.4

Factorize each of the following cubic polynomials by factor theorem.

Q.1 x^3-2x^2-x+2

Let $P(x)=x^3-2x^2-x+2$

Put $x=1$

$P(1)=(1)^3-2(1)^2-(1)+2$

$=1-2-1+2$

$=-3+3=0$

As, R = 0,

So $(x-1)$ is a factor

Put $x=-1$

$P(-1)=(-1)^3-2(-1)^2-(-1)+2$

$=-1-2+1+2$

As R = 0,

So (x+1) is the second factor of p(x).

Put $x=2$

$P(2)=(2)^3-2(2)^2-(2)+2$

$=8-8-2+2$

$=10-10$

$=0$

As R = 0,

So $(x-2)$ is the third factor

Hence $P(x)=x^3-2x^2-x+2$

$=(x-1)(x+1)(x-2)$

Q.2 $x^3-x^2-22x+40$

Sol:

Let $P(x)=x^3-x^2-22x+40$

Put $x=1$

$P(1)=(1)^3-(1)^2-22(1)+40$

$=1-1-22+40$

$=18 \neq 0$

Hence $x-1$ is not a zero of $P(x)$

Put $x=-1$

$P(-1)=(-1)^3-(-1)^2-22(-1)+40$

$=-1-1+22+40$

$=60 \neq 0$

Hence $x=-1$ is not a zero of $P(x)$

Put $x=2$

$P(2)=(2)^3-(2)^2-22(2)+40$

$=8-4-44+40=0$

Hence $x-2$ is a zero of $P(x)$

So $(x-2)$ is a factor

Put $x=-2$

$P(-2)=(-2)^3-(-2)^2-22(-2)+40$

$=-8-4+44+40=72$

Hence $x=-2$ is not a zero of $P(x)$

Put $x=3$

$P(3)=(3)^3-(3)^2-22(3)+40$

$=27-9-66+40$

$=67-75$

$=-8 \neq 0$

Hence $x=3$ is not a zero of $P(x)$

Put $x=-3$

$P(-3)=(-3)^3-(-3)^2-22(-3)+40$

$=-27-9+66+40$

$=106-36$

$=70 \neq 0$

Hence $x=-3$ is not a zero of $P(x)$

Put $x=4$

$P(4)=(4)^3-(4)^2-22(4)+40$

$=64-16-88+40$

$=104-104$

$=0$

Hence $x=4$ is a zero of $P(x)$

So $(x-4)$ is second factor

Put $x=-4$

$P(-4)=(-4)^3-(-4)^2-22(-4)+40$

$=-64-16+88+40$

$=-80+128$

$=48 \neq 0$

So, $x=-4$ is not a zero of $P(x)$

Put $x=5$

$P(5)=(5)^3-(5)^2-22(5)+40$

$=125-25-110+40$

$=165-135$

$=30 \neq 0$

So, $x=5$ is not a zero of $P(x)$

Put $x=-5$

$P(-5)=(-5)^3-(5)^2-22(-5)+40$

$=-125-25+110+40$

$=-150+150$

$=0$

So, $x=-5$ is a zero of $P(x)$

Hence $x+5$ is third factor of $P(x)$

Hence $P(x)=x^3-x^2-22x+40$

$=(x-2)(x-4)(x+5)$

Q.3 $x^3-6x^2+3x+10$

Sol:

Let $P(x)=x^3-6x^2-6x^2+3x+10$

Put $x=1$

$P(1)=(1)^3-6(1)^2+3(1)+10$

$=1-6+3+10$

$=14-6$
$=8\neq 0$

So, $x=1$ is not a zero of $P(x)$

Put $x=-1$

$P(-1)=(-1)^3-6(-1)^2+3(-1)+10$
$=-1-6-3+10$
$=-10+10$
$=0$

So, $x=-1$ is a zero of $P(x)$.

Hence $(x+1)$ is a factor of $P(x)$

Put $x=2$

$P(2)=(2)^3-6(2)^2+3(2)+10$
$=8-24+6+10$
$=24-24$
$=0$

So, $x=2$ is a zero of $P(x)$.

Hence $(x-2)$ is second factor of $P(x)$

Put $x=-2$

$P(-2)=(-2)^3-6(-2)^2+3(-2)+10$
$=-8-24-6+10$
$=-28\neq 0$

So, $x=-2$ is not a zero of $P(x)$

Put $x=3$

$P(3)=(3)^3-6(3)^2+3(3)+10$
$=27-6(9)+9+10$
$=46-54$
$=-8\neq 0$

So, $x=3$ is not a zero of $P(x)$

Put $x=-3$

$P(-3)=(-3)^3-6(-3)^2+3(-3)+10$
$=-27-6(9)-9+10$
$=-90+10$
$=-80\neq 0$

So, $x=-3$ is not a zero of $P(x)$

Put $x=4$

$P(4)=(4)^3-6(4)^2+3(4)+10$
$=64-6(16)+12+10$
$=86-96$
$=-10\neq 0$

So, $x=4$ is not a zero of $P(x)$

Put $x=-4$

$P(-4)=(-4)^3-6(-4)^2+3(-4)+10$
$=-64-6(16)-12+10$
$=-64-96-12+10$
$=-172+10$
$=-162$
$=-162\neq 0$

Put $x=5$

$P(5)=(5)^3-6(5)^2+3(5)+10$
$=125-150+15+10$
$=150-150$
$=0$

So, $x=5$ is a zero of $P(x)$

Hence $(x-5)$ is third factor of $P(x)$

Hence $P(x)=x^3-6x^2+3x+10$
$=(x+1)(x-2)(x-5)$

Q.4 $x^3+x^2-10x+8$

Sol:

Let $P(x)=x^3+x^2-10x+8$

Put $x=1$

$P(1)=(1)^3+(1)^2-10(1)+8$
$=1+1-10+8$
$=0$

So, $x=1$ is a zero of $P(x)$

Hence $(x-1)$ is a factor of $P(x)$

Put $x=-1$

$$P(-1)=(-1)^3+(-1)^2-10(-1)+8$$
$$=-1+1+10+8$$
$$=18\neq 0$$

So, $x=-1$ is not a zero of $P(x)$

Put $x=2$

$$P(2)=(2)^3+(2)^2-10(2)+8$$
$$=8+4-20+8$$
$$=20-20$$
$$=0$$

So, $x=2$ is a zero of $P(x)$

Hence $x-2$ is second factor of $P(x)$

Put $x=-2$

$$P(-2)=(-2)^3+(-2)^2-10(-2)+8$$
$$=-8+4+20+8$$
$$=24\neq 0$$

So, $x=-2$ is not a zero of $P(x)$

Put $x=3$

$$P(3)=(3)^3+(3)^2-10(3)+8$$
$$=27+9-30+8$$
$$=44-30$$
$$=14\neq 0$$

Put $x=-3$

$$P(-3)=(-3)^3+(-3)^2-10(-3)+8$$
$$=-27+9+30+8$$
$$=-27+47$$
$$=20\neq 0$$

So, $x=-3$ is not a zero of $P(x)$

Put $x=4$

$$P(4)=(4)^3+(4)^2-10(4)+8$$
$$=64+16-40+8$$
$$=88-40$$
$$=48\neq 0$$

So, $x=4$ is not a zero of $P(x)$

Put $x=-4$

$$P(-4)=(-4)^3+(-4)^2-10(-4)+8$$
$$=-64+16+40+8$$
$$=-64+64$$
$$=0$$

So, $x=-4$ is a zero of $P(x)$

Hence $x+4$ is third factor of $P(x)$

Hence $P(x)=x^3+x^2-10x+8$

$$=(x-1)(x-2)(x+4)$$

Q.5 x^3-2x^2-5x+6

Sol:

$$P(x)=x^3-2x^2-5x+6$$

Put $x=1$

$$P(1)=(1)^3-2(1)^2-5(1)+6$$
$$=1-2-5+6$$
$$=7-7$$
$$=0$$

So, $x=1$ is a zero of $P(1)$

Hence $x-1$ is a factor of $P(x)$

Put $x=-1$

$$P(-1)=(-1)^3-2(-1)^2-5(-1)+6$$
$$=-1-2+5+6$$
$$=-3+11$$
$$=8\neq 0$$

So, $x=-1$ is not a zero of $P(x)$

Put $x=2$

$$P(2)=(2)^3-2(2)^2-5(2)+6$$

$=8-8-10+6$

$=-4\neq 0$

So, $x=2$ is not a zero of $P(x)$

Put $x=-2$

$P(-2)=(-2)^3-2(-2)^2-5(-2)$

$=-8-8+10+6$

$=0$

So, $x=-2$ is a zero of $P(x)$

Hence $(x+2)$ is second factor of $P(x)$

Put $x=3$

$P(3)=(3)^3-2(3)^2-5(3)+6$

$=27-18-15+6$

$=33-33$

$=0$

So, $x=3$ is a zero of $P(x)$

Hence $(x-3)$ is third factor of $P(x)$

Hence $P(x)=x^3-2x^2-5x+6$

$=(x-1)(x+2)(x-3)$

Q.6 $x^3+5x^2-2x-24$

Sol:

Let $P(x)=x^3+5x^2-2x-24$

Put $x=1$

$P(1)=(1)^3+5(1)^2-2(1)-24$

$=1+5-2-24$

$=6-26$

$=-20\neq 0$

So, $x=1$ is not a zero of $P(x)$

Put $x=-1$

$P(-1)=(-1)^3+5(-1)^2-2(-1)-24$

$=-1+5+2-24$

$=7-25$

$=-18\neq 0$

So, $x=-1$ is not a zero of $P(x)$

Put $x=2$

$P(2)=(2)^3+5(2)^2-2(2)-24$

$=8+20-4-24$

$=28-28$

$=0$

So, $x=2$ is a zero of $P(x)$

Hence $(x-2)$ is a factor of $P(x)$

Put $x=-2$

$P(-2)=(-2)^3+5(-2)^2-2(-2)-24$

$=-8+5(4)+4-24$

$=-32+24$

$=-8\neq 0$

So, $x=-2$ is not a zero of $P(x)$

Put $x=3$

$P(3)=(3)^3+5(3)^2-2(3)-24$

$=27+5(9)-6-24$

$=72-30$

$=42\neq 0$

So, $x=3$ is not a zero of $P(x)$

Put $x=-3$

$P(-3)=(-3)^3+5(-3)^2-2(-3)-24$

$=-27+45+6-24$

$=51-51$

$=0$

So, $x=-3$ is a zero of $P(x)$

Hence $(x+3)$ is second factor of $P(x)$

Put $x=4$

$P(4)=(4)^3+5(4)^2-2(4)-24$

$=64+5(16)-8-24$

$=144-32$

$=112\neq 0$

So, $x=4$ is not a zero of $P(x)$

Put $x=-4$

$P(-4)=(-4)^3+5(-4)^2-2(-4)-24$

$=-64+80+8-24$

$=0$

So, $x=-4$ is a zero of $P(x)$

Hence $(x+4)$ is third factor of $P(x)$

Hence $P(x)=x^3+5x^2-2x-24$

$=(x-2)(x+3)(x+4)$

Q.7 $3x^3-x^2-12x+4$

Sol: $P(x)=3x^3-x^2-12x+4$

Put $x=1$

$P(1)=3(1)^3-(1)^2-12(1)+4$

$=3-1-12+4$

$=7-13$

$=-6\neq 0$

So, $x=1$ is not a zero of $P(x)$

Put $x=-1$

$P(-1)=3(-1)^3-(-1)^2-12(-1)+4$

$=-3-1+12+4$

$=-4+16$

$=12\neq 0$

So, $x=-1$ is not a zero of $P(x)$

Put $x=2$

$P(2)=3(2)^3-(2)^2-12(2)+4$

$=24-4-24+4$

$=28-28$

$=0$

So, $x=2$ is a zero of $P(x)$

Hence $(x-2)$ is a factor of $P(x)$

Put $x=-2$

$P(-2)=3(-2)^3-(-2)^2-12(-2)+4$

$=-24-4+24+4$

$=-28+28$

$=0$

So, $x=-2$ is a zero of $P(x)$

Hence $(x+2)$ is second factor of $P(x)$

Put $3x=1$

$x=\frac{1}{3}$

$P\left(\frac{1}{3}\right)=3\left(\frac{1}{3}\right)^3-\left(\frac{1}{3}\right)^2-12\left(\frac{1}{3}\right)+4$

$=\cancel{3}\left(\frac{1}{27}\right)-\frac{1}{9}-12\left(\frac{1}{3}\right)+4$

$=\frac{1}{9}-\frac{1}{9}-4+4$

$=0$

So, $x=\frac{1}{3}$ is a zero of $P(x)$

Hence $(3x-1)$ is third factor of $P(x)$

Hence $P(x)=3x^3-x^2-12x+4$

$=(x-2)(x+2)(3x-1)$

Q.8 $2x^3+x^2-2x-1$

Let $P(x)=2x^3+x^2-2x-1$

Put $x=1$

$P(1)=2(1)^3+(1)^2-2(1)-1$

$=2+1-2-1$

$=3-3$

$=0$

So, $x=1$ is a zero of $P(x)$

Hence $(x-1)$ is a factor of $P(x)$

Put $x=-1$

$$P(-1)=2(-1)^3+(-1)^2-2(-1)-1$$
$$=-2+1+2-1$$
$$=-1+1$$
$$=0$$

So, $x=-1$ is a zero of $P(x)$

Hence $(x+1)$ is second factor of $P(x)$

Put $2x=1$

$$x=\frac{1}{2}$$

$$P\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^2-2\left(\frac{1}{2}\right)-1$$
$$=2\left(\frac{1}{8}\right)+\left(\frac{1}{2}\right)^2-2\left(\frac{1}{2}\right)-1$$
$$=\frac{1}{4}+\frac{1}{4}-1-1$$
$$=\frac{-3}{2}\neq 0$$

So, $x-2$ is not a zero of $P(x)$

Put $x=\frac{-1}{2}$

$$P\left(\frac{-1}{2}\right)=2\left(\frac{-1}{2}\right)^3+\left(\frac{-1}{2}\right)^2-2\left(\frac{-1}{2}\right)-1$$
$$=2\left(-\frac{1}{8}\right)+\frac{1}{4}+1-1$$
$$=-\frac{1}{4}+\frac{1}{4}+1-1$$
$$=0$$

So, $x=\frac{-1}{2}$ is a zero of $P(x)$

Hence $2x+1$ is third factor of $P(x)$

Hence $P(x)=2x^3+x^2-2x-1$
$$=(x-1)(x+1)(2x+1)$$

Objective

1. The factor of x^2-5x+6 are: ____
   (a) $x+1, x-6$ (b) $x-2, x-3$
   (c) $x+6, x-1$ (d) $x+2, x+3$
2. Factors of $8x^3+27y^3$ are:____
   (a) $(2x+3y)(4x^2-9y^2)$
   (b) $(2x-3y)(4x^2-9y^2)$
   (c) $(2x+3y)(4x^2-6xy+9y^2)$
   (d) $(2x-3y)(4x^2+6xy+9y^2)$
3. Factors of $3x^2-x-2$ are:
   (a) $(x+1)(3x-2)$ (b) $(x+1)(3x+2)$
   (c) $(x-1)(3x-2)$ (d) $(x-1)(3x+2)$
4. Factors of a^4-4b^4 are: ____
   (a) $(a-b)(a+b)(a^2+4b^2)$
   (b) $(a^2-2b^2)(a^2+2b^2)$
   (c) $(a-b)(a+b)(a^2-4b^2)$
   (d) $(a-2b)(a^2+2b^2)$
5. What will be added to complete the square of $9a^2-12ab$?____
   (a) $-16b^2$ (b) $16b^2$
   (c) $4b^2$ (d) $-4b^2$
6. Find m so that x^2+4x+m is a complete square:
   (a) 8 (b) −8
   (c) 4 (d) 16
7. Factors of $5x^2-17xy-12y^2$ are____
   (a) $(x+4y)(5x+3y)$
   (b) $(x-4y)(5x-3y)$
   (c) $(x-4y)(5x+3y)$
   (d) $(5x-4y)(x+3y)$

8. Factors of $27x^3 - \frac{1}{x^3}$ are___

(a) $\left(3x - \frac{1}{x}\right)\left(9x^2 + 3 + \frac{1}{x^2}\right)$

(b) $\left(3x + \frac{1}{x}\right)\left(9x^2 + 3 + \frac{1}{x^2}\right)$

(c) $\left(3x - \frac{1}{x}\right)\left(9x^2 - 3 + \frac{1}{x^2}\right)$

(d) $\left(3x + \frac{1}{x}\right)\left(9x^2 - 3 + \frac{1}{x^2}\right)$

9. If $x - 2$ is a factor of $p(x) = x^2+2kx+8$, then K = ___

(a) −3 (b) 3

(c) 4 (d) 5

10. $4a^2+4ab+(\ldots..)$ is a complete square

(a) b^2 (b) 2b

(c) a^2 (d) $4b^2$

11. $\frac{x^2}{y^2} - 2 + \frac{y^2}{x^2} = \ldots\ldots\ldots$

(a) $\left(\frac{x}{y} - \frac{y}{x}\right)^2$ (b) $\left(\frac{x}{y} + \frac{y}{x}\right)^2$

(c) $\left(\frac{x}{y} - \frac{y}{x}\right)^3$ (d) $\left(\frac{x}{y} + \frac{y}{x}\right)^3$

12. $(x+y)(x^2 - xy + y^2) =$ ______

(a) $x^3 - y^3$ (b) $x^3 + y^3$

(c) $(x+y)^3$ (d) $(x - y)^3$

13. Factors of $x^4 - 16$ is ___

(a) $(x-2)^2$

(b) $(x-2)(x+2)(x^2+4)$

(c) $(x-2)(x+2)$

(d) $(x+2)^2$

14. Factors of $3x - 3a + xy - ay$.

(a) $(3+y)(x-a)$

(b) $(3-y)(x+a)$

(c) $(3-y)(x-a)$

(d) $(3+y)(x+a)$

15. Factors of $pqr + qr^2 - pr^2 - r^3$ is:

(a) $r(p+r)(q-r)$ (b) $r(p-r)(q+r)$

(c) $r(p-r)(q-r)$ (d) $r(p+r)(q+r)$

Answer Key

1.	b	2.	c	3.	d	4.	b	5.	c
6.	c	7.	c	8.	a	9.	a	10.	a
11.	a	12.	b	13.	b	14.	a	15.	a

Unit 06

ALGEBRAIC MANIPULATION

Highest Common Factor (H.C.F.)

If two or more algebraic expressions are given then their common factor of highest power is called the H.C.F of the expressions.

Least Common Multiple (L.C.M)

If an algebraic expression $p(x)$ is exactly divisible by two or more expressions, then p(x) is called the Common Multiple of the given expressions. The Least Common Multiple (L.C.M) is the product of common factors together with non-common factors of the given expressions.

Finding H.C.F

We can find H.C.F of given expressions by the following two methods.

(i) By Factorization

(ii) By division

H.C.F. by Factorization

Example

Find the H.C.F of the following polynomials.

$$x^2-4, x^2+4x+4, 2x^2+x-6$$

Solution

By factorization,

$$x^2-4=(x+2)(x-2)$$

$$x^2+4x+4=(x+2)^2=(x+2)(x+2)$$

$$2x^2+x-6=2x^2+4x-3x-6=2x(x+2)-3(x+2)$$
$$=(x+2)(2x-3)$$

Common factors $= x+2$

H.C.F $= x+2$

H.C.F. by Division

Example

Use division method to find the H.C.F. of the polynomials

$$p(x)=x^3-7x^2+14x-8 \quad and$$
$$q(x)=x^3-7x+6$$

Solution

$$\begin{array}{r|l}
 & \quad 1 \\ \hline
x^3-7x+6 & x^3-7x^2+14x-8 \\
 & \underset{-}{+x^3} \qquad \underset{+}{-7x} \; \underset{-}{+6} \\ \hline
 & -7x^2+21x-14
\end{array}$$

Here the remainder can be factorized as

$$-7x^2+21x-14=-7(x^2-3x+2)$$

We ignore -7 because it is not common to both the given polynomials and consider x^2-3x+2.

$$\begin{array}{r|l}
 & \quad x+3 \\ \hline
x^2-3x+2 & x^3+0x^2-7x+6 \\
 & \underset{-}{+x^3}\,\underset{+}{-3x^2}\,\underset{-}{+2x} \\ \hline
 & 3x^2-9x+6 \\
 & \underset{-}{3x^2}\,\underset{+}{-9x}\,\underset{-}{+6} \\ \hline
 & \quad 0
\end{array}$$

Hence H.C. F of $p(x)\, and\, q(x)$ is x^2-3x+2

Example

Find the L.C.M of $p(x)=12(x^3-y^3)$ and $q(x)=8(x^3-xy^2)$

Solution

By prime factorization of the given expressions, we have

$p(x)=12(x^3-y^3)=2^2\times3\times(x-y)(x^2+xy+y^2)$ and

$q(x)=8(x^3-xy^2)=8x(x^2-y^2)=2^3x(x+y)(x-y)$ Hence L.C.M. of $p(x)$ and $q(x)$,

$2^3\times3\times x(x+y)(x-y)(x^2+xy+y^2)=24x(x+y)(x^3-y^3)$

Relation between H.C.F and L.C.M

Example

By factorization, find (i) H.C.F (ii) L.C.M of $p(x)=12(x^5-x^4)$ and $q(x)=8(x^4-3x^3+3x^2)$. Establish a relation between $p(x)$, $q(x)$ and H.C.F and L.C.M of the expressions $p(x)$ and $q(x)$.

Solution

Firstly, let us factorize completely the given expressions $p(x)$ and $q(x)$ into irreducible factors. We have

$p(x)=12(x^5-x^4)=12x^4(x-1)=2^2\times3\times x^4(x-1)$ and

$q(x) = 8(x^4-3x^3+2x^2)=8x^2(x^2-3x+2)=2^3x^2(x-1)(x-2)$

H.C.F. of $p(x)$ and $q(x)$ $=$ $2^2x^2(x-1)=4x^2(x-1)$

L.C.M of $p(x)$ and $q(x)$ $=$ $2^3\times3\times x^4(x-1)(x-2)$

Now $p(x)\,q(x) = 12x^4(x-1)\times 8x^2(x-1)(x-2)$

$= 96x^6(x-1)^2(x-2)$ ....(i)

and (L.C.M) (H.C.F)

$= [2^3\times3\times x^4(x-1)(x-2)]\,[4x^2(x-1)]$

$= [24x^4(x-1)(x-2)]\,[4x^2(x-1)]$

$= 96x^4(x-1)^2(x-2)$ .... (ii)

From (i) and (ii)

L.C.M x H.C.F = P(x) x q(x)

Note

(1) $\text{L.C.M}=\dfrac{p(x)\times q(x)}{\text{H.C.F}}$ or

$\text{H.C.F}=\dfrac{p(x)\times q(x)}{\text{L.C.M}}$

(2) If L.C.M, H.C.F and one of $p(x)$ or $q(x)$ are known, then

$$p(x) = \frac{\text{L.C.M}\times\text{H.C.F}}{q(x)}$$

$$q(x) = \frac{\text{L.C.M}\times\text{H.C.F}}{p(x)}$$

Example

Find H.C.F of the polynomials,

$p(x) \quad = \quad 20(2x^3+3x^2-2x)$

$q(x) \quad = \quad 9(5x^4+40x)$

Then using the above formula (I) find the L.C.M of $p(x)$ and $q(x)$.

Solution

We have

$p(x) = 20(2x^3+3x^2-2x) = 20x(2x^2+3x-2)$

$= 20x(2x^2+4x-x-2) = 20x[2x(x+2)-(x+2)] = 20x(x+2)(2x-1) = 2^2\times5\times x(x+2)(2x-1)$

$q(x) \quad = \quad 9(5x^4+40x) = 45x(x^3+8) = 45x\left[\left(x^3\right)+(2)^3\right]$

$= \quad 45x(x+2)(x^2-2x+4) = 5\times3^2\times x(x+2)(x^2-2x+4)$ Thus H.C.F of $p(x)$ and $q(x)$ is:

$= \quad 5x(x+2)$

Now, using the formula $\quad$ L.C.M. $= \dfrac{p(x)\times q(x)}{\text{H.C.F}}$

We obtain

$$\text{L.C.M.} = \frac{2^2\times5\times x(x+2)(2x-1)\times5\times3^2\times x(x+2)(x^2-2x+4)}{5x(x+2)}$$

$$= 4\times5\times9\times x(x+2)(2x-1)(x^2-2x+4)$$

$$= 180x(x+2)(2x-1)(x^2-2x+4)$$

Example

Find the L.C.M of

$p(x) = 6x^3-7x^2-27x+8$ and

$q(x) = 6x^3+17x^2+9x-4$

Solution

We have, by long division,

$$\begin{array}{r|l} & \quad 1 \\ 6x^3-7x^2-27x+8 & 6x^3+17x^2+9x-4 \\ & 6x^3-7x^2-27x+8 \\ & -\quad+\quad+\quad- \\ \hline & \quad 24x^2+36x-12 \end{array}$$

But the remainder $24x^2+36x-12$

$= 12(2x^2+3x-1)$

Thus, ignoring 12, we have

$$\begin{array}{r|l} & \quad 3x-8 \\ 2x^2+3x-1 & 6x^3-7x^2-27x+8 \\ & 6x^3+9x^2-3x \\ & -\quad-\quad+ \\ \hline & -16x^2-24x+8 \\ & -16x^2-24x+8 \\ & +\quad+\quad- \\ \hline & \quad 0 \end{array}$$

Hence H.C.F of $p(x)$ and $q(x)$ is

$= 2x^2+3x-1$

$x^2+6x-27=x^2-3x+9x-27$

$=x(x-3)+9(x-3)$

$=(x-3)(x+9)$ ....(ii)

$2x^2-18=2(x^2-9)$

$=2\left[(x)^2-(3)^2\right]$

$=2(x+3)(x-3)$ ....(iii)

From (i), (ii) and (iii)

Common factors $=(x-3)$

$HCF=x-3$

iii) $x^3-2x^2+x,\ x^2+2x-3,\ x^2+3x-4$

Sol: By factorization

$x^3-2x^2+x=x(x^2-2x+1)$

$=x(x^2-x-x+1)$

$=x[x(x-1)-1(x-1)]$

$=x(x-1)(x-1)$ ....(i)

$x^2+2x-3=x^2-x+3x-3$

$=x(x-1)+3(x-1)$

$=(x-1)(x+3)$ ....(ii)

$x^2+3x-4=x^2-x+4x-4$

$=x(x-1)+4(x-1)$

$=(x-1)(x+4)$ ....(iii)

From (i), (ii) and (iii)

Common factors: $x-1$

HCF $=x-1$

iv) $18(x^3+9x^2+8x),\ 24(x^2-3x+2)$

Sol: By factorization

$18(x^3-9x^2+8x)=18x(x^2-9x+8)$

$=18x(x^2-x-8x+8)$

$=18x[x(x-1)-8(x-1)]$

$=2\times3\times3\ x(x-1)(x-8)$ ....(i)

$24(x^2-3x+2)=$

$24(x^2-x-2x+2)$

$=2\times2\times2\times3[x(x-1)-2(x-1)]$

$=2\times2\times2\times3(x-1)(x-2)$....(ii)

From (i) and (ii)

HCF $=2\times3(x-1)$

$=6(x-1)$

v) $36(3x^4+5x^3-2x^2),\ 54(27x^4-x)$

Sol: By factorization

$36(3x^4+5x^3-2x^2)=36x^2(3x^2+5x-2)$

$=36x^2(3x^2+6x-x-2)$

$=36x^2[3x(x+2)-1(x+2)]$

$=2\times2\times3\times3x.x(x+2)(3x-1)$ ....(i)

$54(27x^4-x)=54x(27x^3-1)$

$=54x\left[(3x)^3-(1)^3\right]$

$=54x(3x-1)\left[(3x)^2+(3x)(1)+(1)^2\right]$

$=2\times3\times3\times3x(3x-1)(9x^2+3x+1)$ .....(ii)

From (i) and (ii)

Common factors $=2,3,3,x,(3x-1)$

HCF $=2\times3\times3x(3x-1)$

$=18x(3x-1)$

Q3. Find the H.C.F of the following by division methal.

i) $p(x)=x^3+3x^2-16x+12,\ q(x)=x^3+x^2-10x+8$

Sol:
$$\begin{array}{r|l} & \quad\quad\quad\quad\quad 1 \\ x^3+x^2-10x+8 & x^3+3x^2-16x+12 \\ & \underline{-x^3\pm x^2\mp10x\pm8} \\ & \quad 2x^2-6x+4 \end{array}$$

Dividing remainder by 2

$$x^2-3x+2$$

$$x^2-3x+2\overline{\big)\,x^3+x^2-10x+8\,}\quad x+4$$

$$\cancel{-x^3}\mp3x^2\pm2x$$

$$\cancel{4x^2}-\cancel{12x}+\cancel{8}$$

$$\cancel{-4x^2}-\cancel{12x}\pm\cancel{8}$$

$$0$$

Hence HCF $= x^2-3x+2$

ii) $P(x)=x^4+x^3-2x^2+x-3,$

$q(x)=5x^3+3x^2-17x+6$

$$5x^3+3x^2-17x+6\overline{\big)\,x^4+x^3-2x^2+x-3\,}\quad x+2$$

$$\times5 \quad \text{(Multiplying by 5)}$$

$$\cancel{5x^4}+5x^3-10x^2+5x-15$$

$$\cancel{-5x^4}\pm3x^3\mp17x^2\pm6x$$

$$2x^3+7x^2-x-15$$

$$\times5 \quad \text{(Multiplying by 5)}$$

$$\cancel{10x^3}+35x^2-5x-75$$

$$\cancel{-10x^3}\pm6x^2\mp34x\pm12$$

$$29x^2+29x-87$$

Divided by 29

$$x^2+x-3$$

$$x^2+x-3\overline{\big)\,5x^3+3x^2-17x+6\,}\quad 5x-2$$

$$\cancel{-5x^3}\pm5x^2\mp15x$$

$$\cancel{-2x^2}-\cancel{2x}+\cancel{6}$$

$$\cancel{\mp2x^2}\mp\cancel{2x}\pm\cancel{6}$$

$$0$$

Hence H.C.F $= x^2+x-3$

iii) $p(x)=2x^5-4x^4-6x,$

$q(x)=x^5+x^4-3x^3-3x^2$

$$x^5+x^4-3x^3-3x^2\overline{\big)\,2x^5-4x^4-6x\,}\quad 2$$

$$\cancel{-2x^5}\pm2x^4\quad\mp6x^3\mp6x^2$$

$$-6x^4+6x^3+6x^2-6x$$

Dividing by – 6

$$x^4-x^3-x^2+x$$

$$x^4-x^3-x^2+x\overline{\big)\,x^5+x^4-3x^3-3x^2\,}\quad x+2$$

$$\cancel{-x^5}\mp x^4\mp x^3\pm x^2$$

$$\cancel{2x^4}-\cancel{2x^3}-4x^2$$

$$\cancel{-2x^4}\mp\cancel{2x^3}\mp2x^2\pm2x$$

$$-2x^2-2x$$

Dividing by – 2

$$x^2+x$$

$$x^2+x\overline{\big)\,x^4-x^3-x^2+x\,}\quad x^2-2x+1$$

$$\cancel{-x^4}\pm x^3$$

$$\cancel{-2x^3}-x^2+x$$

$$\cancel{\mp2x^3}\mp2x^2$$

$$\cancel{x^2}+\cancel{x}$$

$$\pm\cancel{x^2}\pm\cancel{x}$$

$$0$$

Hence H.C.F $= x^2+x=x(x+1)$

Q4. Find the L.C.M of the following expressions:

i) $39x^7y^3z$ and $91x^5y^6z^7$

Sol: By factorization

$39x^7y^3z=13\times3\,x.x.x.x.x.x.x.y.y.y.z$

$91x^5y^6z^7=13\times7\,x.x.x.x.x.y.y.y.y.y.y.z.z.z.z.z.z.z$

Hence L.C.M =

$13\times3\times7.\ x.x.x.x.x.x.x.\ y.y.y.y.y.y.\ z.z.z.z.z.z.z$

$=273x^7y^6z^7$

ii) $102xy^2z$, $85x^2yz$ and $187xyz^2$

Sol: By factorization

$102xy^2z$	=	$2\times3\times17x.y.y.z$
$85x^2yz$	=	$5\times17x.x.y.z$
$187xyz^2$	=	$11\times17x.y.z.z$

Hence L.C.M $= 17\times 11\times 5\times 3\times 2.x.x.y.y.z.z$

$= 5610x^2y^2z^2$

Q5. Find the L.C.M of the following expressions by factorization:

i) $x^2-25x+100$ and x^2-x-20

Sol: By factorization

$x^2-25x+100 = x^2-5x-20x+100$

$= x(x-5)-20(x-5)$

$= (x-5)(x-20)$.............(i)

$x^2-x-20 = x^2-5x+4x-20$

$= x(x-5)+4(x-5)$

$= (x-5)(x+4)$ ....(ii)

From (i) and (ii)

L.C.M $= (x-5)(x-20)(x+4)$

ii) $x^2+4x+4,\ x^2-4,\ 2x^2+x-6$

Sol: By factorization

$x^2+4x+4 = x^2+2x+2x+4$

$= x(x+2)+2(x+2)$

$= (x+2)(x+2)$ ....(i)

$x^2-4 = (x)^2-(2)^2$

$= (x+2)(x-2)$ ....(ii)

$2x^2+x-6 = 2x^2+4x-3x-6$

$= 2x(x+2)-3(x+2)$

$= (x+2)(2x-3)$ ....(iii)

From (i), (ii) and (iii)

LCM $= (x+2)(x+2)(x-2)(2x-3)$

$= (x+2)^2(x-2)(2x-3)$

iii) $2(x^4-y^4),\ 3(x^3+2x^2y-xy^2-2y^3)$

Sol: By factorization

$2(x^4-y^4) = 2\left[(x^2)^2-(y^2)^2\right]$

$= 2(x^2+y^2)(x^2-y^2)$

$= 2(x^2+y^2)(x+y)(x-y)$ ....(i)

$3(x^3+2x^2y-xy^2-2y^3) = 3\left[x^2(x+2y)-y^2(x+2y)\right]$

$= 3(x+2y)(x^2-y^2)$

$= 3(x+2y)(x+y)(x-y)$ ....(ii)

From (i) & (ii)

L.C.M =

$2\times 3(x+y)(x-y)(x^2+y^2)(x+2y)$

$= 6(x^4-y^4)(x+2y)$

iv) $4(x^4-1), 6(x^3-x^2-x+1)$

Sol: By factorization

$4(x^4-1) = 4\left[(x^2)^2-(1)^2\right]$

$= 4(x^2+1)(x^2-1)$

$= 2\times 2(x^2+1)\left[(x)^2-(1)^2\right]$

$= 2\times 2(x^2+1)(x+1)(x-1)$ ....(i)

$6(x^3-x^2-x+1) = 6\left[x^2(x-1)-1(x-1)\right]$

$= 6(x-1)(x^2-1) = 2\times 3(x-1)\left[(x)^2-(1)^2\right]$

$= 2\times 3(x-1)(x-1)(x+1)$ ....(ii)

From (i) & (ii)

LCM= $2\times 2\times 3(x+1)(x-1)(x^2+1)(x-1)$

$= 12(x^4-1)(x-1)$

Q6. For what value of k is $(x+4)$, the H.C.F of $x^2+x-(2k+2)$ and $2x^2+kx-12$?

Sol: $k = ?$

$p(x) = x^2+x-(2k+2)$ and

$q(x) = 2x^2+kx-12$

As given that $x+4$ is HCF, so $p(x)$ and $q(x)$ will be exactly divisible by $(x+4)$

$$\begin{array}{r|l} & x-3 \\ \hline x+4 & \cancel{x^2}+x-(2k+2) \\ & \cancel{\pm x^2}\pm 4x \\ & \cancel{-3x}-(2k+2) \\ & \cancel{\mp 3x}\mp 12 \\ & 12-(2k+2) \end{array}$$

$= 12-2k-2$

$= 10-2k$

As $p(x)$ is exactly divisible by $x+4$, so,

$10-2k=0$

$10=2k$

$\frac{10}{2}=k$

$k=5$

Q7. If $(x+3)(x-2)$ is the H.C.F of $p(x)=(x+3)(2x^2-3x+k)$ and $q(x)=(x-2)(3x^2+7x-l)$, find k and l.

Sol: $k=?$ and $l=?$

As $(x+3)(x-2)$ is the H.C.F, so $p(x)$ and $q(x)$ will be exactly divisible by $(x+3)(x-2)$ i.e., $\frac{p(x)}{HCF}$ has remainder zero.

$$\frac{\cancel{(x+3)}(2x^2-3x+k)}{\cancel{(x+3)}(x-2)}=\frac{2x^2-3x+k}{x-2}$$

i.e

$$\begin{array}{r|l} & 2x+1 \\ \hline x-2 & 2x^2-3x+k \\ & \pm 2x^2 \mp 4x \\ & x+k \\ & \pm x \mp 2 \\ & k+2 \end{array}$$

As remainder $=0$, then

$k+2=0$

$\boxed{k=-2}$

and $\frac{q(x)}{HCF}$ has zero remainder

$$\frac{\cancel{(x-2)}(3x^2+7x-l)}{(x+3)\cancel{(x-2)}}=\frac{3x^2+7x-l}{x+3}$$

$$\begin{array}{r|l} & 3x-2 \\ \hline x+3 & \cancel{3x^2}+7x-l \\ & \cancel{\pm 3x^2}\pm 9x \\ & \cancel{-2x}-l \\ & \cancel{\mp 2x}\mp 6 \\ & -l+6 \end{array}$$

As remainder $=0$

$-l+6=0$

$-l=-6$

$\Rightarrow \boxed{l=6}$

Q8. The LCM and HCF of two polynomials $p(x)$ and $q(x)$ are $2(x^4-1)$ and $(x+1)$ (x^2+1) respectively. If $p(x)=x^3+x+1$, find $q(x)$.

Sol: LCM $=2(x^4-1)$,

HCF $=(x+1)(x^2+1)$

$p(x)=x^3+x^2+x+1,\ q(x)=?$

As $p(x)\times q(x)=(LCM)\times(HCF)$

$$q(x)=\frac{(LCM)\times(HCF)}{p(x)}$$

$$=\frac{2(x^4-1)\times(x+1)(x^2+1)}{x^3+x^2+x+1}$$

$$=\frac{2(x^4-1)\cancel{(x^3+x^2+x+1)}}{\cancel{x^3+x^2+x+1}}$$

$q(x)=2(x^4-1)$

Q9. Let $p(x)=10(x^2-9)(x^2-3x+2)$ and $q(x)=10x(x+3)(x-1)^2$. If the H.C.F. of $p(x),q(x)$ is $10(x+3)(x-1)$, find their L.C.M.

Sol: $p(x)=10(x^2-9)(x^2-3x+2)$,

$q(x)=10x(x+3)(x-1)^2$

H.C.F. $=10(x+3)(x-1)$, L.C.M = ?

As $(L.C.M)\times(H.C.F)=p(x)\times q(x)$

$$\text{L.C.M.}=\frac{p(x)\times q(x)}{H.C.F}$$

$$=\frac{\cancel{10}(x^2-9)(x^2-3x+2)\times 10x(x+3)(x-1)^2}{\cancel{10}(x+3)(x-1)}$$

$$=\frac{(x^2-9)(x^2-3x+2)\times 10x\cancel{(x+3)}\cancel{(x-1)}(x-1)}{\cancel{(x+3)}\cancel{(x-1)}}$$

$=10x(x-1)(x^2-9)(x^2-3x+2)$

$=10x(x-1)(x^2-9)(x^2-x-2x+2)$

$=10x(x-1)(x^2-9)[x(x-1)-2(x-1)]$

$=10x(x-1)(x^2-9)(x-1)(x-2)$

$=10x(x-1)^2(x^2-9)(x-2)$

Q10. Let the product of L.C.M and H.C.F of two polynomials be $(x+3)^2(x-2)(x+5)$. If one polynomial is $(x+3)(x-2)$ and the second polynomial is $x^2+kx+15$, find the value of k.

Sol: $k=?$

Product of L.C.M. & H.C.F is

$LCM\times HCF=(x+3)^2(x-2)(x+5)$

$p(x)=(x+3)(x-2)$

$q(x)=x^2+kx+15$

As $p(x)\times q(x)=LCM\times HCF$

$(x+3)(x-2)(x^2+kx+15)$

$=(x+3)^2(x-2)(x+5)$

$$x^2+kx+15=\frac{(x+3)\cancel{(x+3)}\cancel{(x-2)}(x+5)}{\cancel{(x+3)}\cancel{(x-2)}}$$

$x^2+kx+15=(x+3)(x+5)$

$x^2+kx+15=x^2+3x+5x+15$

$x^2+kx+15=x^2+8x+15$

Comparing co-efficient of 'x'

$\Rightarrow kx=8x$

$\boxed{k=8}$

Q11. Waqas wishes to distribute 128 bananas and also 176 apples equally among a certain number of children. Find the highest number of the Children. Who can get the fruit in this way?

Sol: No. of bananas = 128

No. of apples = 176

Highest no. of children who get the fruit in this way is H.C.F.

So No. of bananas =

$2\times2\times2\times2\times2\times2\times2$

No. of apples =

$2\times2\times2\times2\times11$

Hence required no. of children =

$2\times2\times2\times2=16$

Example

Simplify

$$\frac{x+3}{x^2-3x+2}+\frac{x+2}{x^2-4x+3}+\frac{x+1}{x^2-5x+6},\ x\neq 1,2,3$$

Solution

$$\frac{x+3}{x^2-3x+2}+\frac{x+2}{x^2-4x+3}+\frac{x+1}{x^2-5x+6}$$

$$=\frac{x+3}{x^2-2x-x+2}+\frac{x+2}{x^2-3x-x+3}+\frac{x+1}{x^2-3x-2x+6}$$

$$=\frac{x+3}{x(x-2)-1(x-2)}+\frac{x+2}{x(x-3)-1(x-3)}+\frac{x+1}{x(x-3)-2(x-3)}$$

$$=\frac{x+3}{(x-2)(x-1)}+\frac{x+2}{(x-3)(x-1)}+\frac{x+1}{(x-3)(x-2)}$$

$$=\frac{(x+3)(x-3)+(x+2)(x-2)+(x+1)(x-1)}{(x-1)(x-2)(x-3)}$$

$$=\frac{x^2-9+x^2-4+x^2-1}{(x-1)(x-2)(x-3)}$$

$$=\frac{3x^2-14}{(x-1)(x-2)(x-3)}$$

Example

Express the product $\frac{x^3-8}{x^2-4}\times\frac{x^2+6x+8}{x^2-2x+1}$ as an algebraic expression reduced lowest forms $x\neq 2,-2,1$

Solution

By factorizing completely, we have

$$\frac{x^3-8}{x^2-4}\times\frac{x^2+6x+8}{x^2-2x+1}$$

$$=\frac{(x-2)(x^2+2x+4)\times(x+2)(x+4)}{(x-2)(x+2)\times(x-1)^2}\quad ..(i)$$

Now the factors of numerator are $(x-2),(x^2+2x+4),(x+2)$ and $(x+4)$ and the factors of denominator are

$(x-2),(x+2)$ and $(x-1)^2$.

Therefore, their H.C.F. is $(x-2)\times(x+2)$

By cancelling H.C.F i.e., $(x-2)\times(x+2)$ from (i), we get the simplified form of given product as the fraction $\frac{(x^2+2x+4)(x+4)}{(x-1)^2}$

Example

Divide $\frac{x^2+x+1}{x^2-9}$ by $\frac{x^3-1}{x^2-4x+3}$ and simplify by reducing to lowest forms.

Solution

We have $\frac{x^2+x+1}{x^2-9}\div\frac{x^3-1}{x^2-4x+3}$

$$=\frac{(x^2+x+1)}{(x^2-9)}\times\frac{(x^2-4x+3)}{(x^3-1)}$$

$$=\frac{(x^2+x+1)(x^2-x-3x+3)}{(x^2-9)(x^3-1)}$$

$$=\frac{(x^2+x+1)[x(x-1)-3(x-1)]}{(x+3)(x-3)(x-1)(x^2+x+1)}$$

$$=\frac{(x^2+x+1)(x-3)(x-1)}{(x+3)(x-3)(x-1)(x^2+x+1)}=\frac{1}{x+3},\ x\neq-3$$

Exercise 6.2

Simplify each of the following as a rational expression.

Q1. $\frac{x^2-x-6}{x^2-9}+\frac{x^2+2x-24}{x^2-x-12}$

$$=\frac{x^2-3x+2x-6}{(x)^2-(3)^2}+\frac{x^2+6x-4x-24}{x^2+3x-4x-12}$$

$$=\frac{x(x-3)+2(x-3)}{(x+3)(x-3)}+\frac{x(x+6)-4(x+6)}{x(x+3)-4(x+3)}$$

$$=\frac{(x-3)(x+2)}{(x+3)(x-3)}+\frac{(x+6)(x-4)}{(x+3)(x-4)}$$

$$=\frac{x+2}{x+3}+\frac{x+6}{x+3}=\frac{x+2+x+6}{x+3}$$

$$=\frac{2x+8}{x+3}$$
$$=\frac{2(x+4)}{x+3}$$

Q2. $\left[\frac{x+1}{x-1}-\frac{x-1}{x+1}-\frac{4x}{x^2+1}\right]+\frac{4x}{x^4-1}$

$$=\left[\frac{(x+1)^2-(x-1)^2}{(x-1)(x+1)}-\frac{4x}{x^2+1}\right]+\frac{4x}{x^4-1}$$
$$=\left[\frac{(x^2+2x+1)-(x^2-2x+1)}{(x)^2-(1)^2}-\frac{4x}{x^2+1}\right]+\frac{4x}{x^4-1}$$
$$=\left[\frac{x^2+2x+1-x^2+2x-1}{x^2-1}-\frac{4x}{x^2+1}\right]+\frac{4x}{x^4-1}$$
$$=\left[\frac{4x}{x^2-1}-\frac{4x}{x^2+1}\right]+\frac{4x}{x^4-1}$$
$$=\left[\frac{4x(x^2+1)-4x(x^2-1)}{(x^2-1)(x^2+1)}\right]+\frac{4x}{x^4-1}$$
$$=\frac{4x^3+4x-4x^3+4x}{(x^2)^2-(1)^2}+\frac{4x}{x^4-1}$$
$$=\frac{8x}{x^4-1}+\frac{4x}{x^4-1}$$
$$=\frac{8x+4x}{x^4-1}$$
$$=\frac{12x}{x^4-1}$$

Q3. $\frac{1}{x^2-8x+15}+\frac{1}{x^2-4x+3}-\frac{2}{x^2-6x+5}$

$$=\frac{1}{x^2-3x-5x+15}+\frac{1}{x^2-3x-x+3}-\frac{2}{x^2-5x-x+5}$$
$$=\frac{1}{x(x-3)-5(x-3)}+\frac{1}{x(x-3)-1(x-3)}-\frac{2}{x(x-5)-1(x-5)}$$
$$=\frac{1}{(x-3)(x-5)}+\frac{1}{(x-3)(x-1)}-\frac{2}{(x-5)(x-1)}$$
$$=\frac{x-1+x-5-2(x-3)}{(x-1)(x-3)(x-5)}$$
$$=\frac{x-1+x-5-2x+6}{(x-1)(x-3)(x-5)}$$
$$=\frac{2x-6-2x+6}{(x-1)(x-3)(x-5)}$$
$$=\frac{0}{(x-1)(x-3)(x-5)}$$
$$=0$$

Q4. $\frac{(x+2)(x+3)}{x^2-9}+\frac{(x+2)(2x^2-32)}{(x-4)(x^2-x-6)}$

$$=\frac{(x+2)(x+3)}{(x)^2-(3)^2}+\frac{(x+2).2(x^2-16)}{(x-4)(x^2+2x-3x-6)}$$
$$=\frac{(x+2)(x+3)}{(x-3)(x+3)}+\frac{2(x+2)\left[(x)^2-(4)^2\right]}{(x-4)(x^2+2x-3x-6)}$$
$$=\frac{(x+2)}{x-3}+\frac{2(x+2)(x+4)(x-4)}{(x-4)(x+2)(x-3)}$$
$$=\frac{x+2}{x-3}+\frac{2x+8}{x-3}$$
$$=\frac{x+2+2x+8}{x-3}$$
$$=\frac{3x+10}{x-3}$$

Q5. $\frac{x+3}{2x^2+9x+9}+\frac{1}{2(2x-3)}-\frac{4x}{4x^2-9}$

$$=\frac{x+3}{2x^2+6x+3x+9}+\frac{1}{2(2x-3)}-\frac{4x}{(2x)^2-(3)^2}$$
$$=\frac{x+3}{2x(x+3)+3(x+3)}+\frac{1}{2(2x-3)}-\frac{4x}{(2x+3)(2x-3)}$$

$$=\frac{\cancel{x+3}}{\cancel{(x+3)}(2x+3)}+\frac{1}{2(2x-3)}-\frac{4x}{(2x+3)(2x-3)}$$

$$=\frac{1}{2x+3}+\frac{1}{2(2x-3)}-\frac{4x}{(2x+3)(2x-3)}$$

$$=\frac{2(2x-3)+2x+3-2(4x)}{2(2x+3)(2x-3)}$$

$$=\frac{4x-6+2x+3-8x}{2(2x+3)(2x-3)}$$

$$=\frac{-2x-3}{2(2x+3)(2x-3)}$$

$$=\frac{-1\cancel{(2x+3)}}{2\cancel{(2x+3)}(2x-3)}$$

$$=\frac{-1}{2(2x-3)}$$

$$=\frac{1}{2(3-2x)}$$

Q6. $A-\frac{1}{A}$, **where** $A=\frac{a+1}{a-1}$

so $\frac{1}{A}=\frac{a-1}{a+1}$

Now $A-\frac{1}{A}=\frac{a+1}{a-1}-\frac{a-1}{a+1}$

$$=\frac{(a+1)^2-(a-1)^2}{(a-1)(a+1)}$$

$$=\frac{(a^2+2a+1)-(a^2-2a+1)}{(a)^2-(1)^2}$$

$$=\frac{\cancel{a^2}+2a+\cancel{1}-\cancel{a^2}+2a-\cancel{1}}{a^2-1}$$

$$=\frac{4a}{a^2-1}$$

Q7. $\left[\frac{x-1}{x-2}+\frac{2}{2-x}\right]-\left[\frac{x+1}{x+2}+\frac{4}{4-x^2}\right]$

$$=\left[\frac{-(x-1)}{2-x}+\frac{2}{2-x}\right]-\left[\frac{x+1}{x+2}+\frac{4}{(2)^2-(x)^2}\right]$$

$$=\left[-\frac{(x-1)}{2-x}+\frac{2}{2-x}\right]-\left[\frac{x+1}{x+2}+\frac{4}{(2+x)(2-x)}\right]$$

$$=\left[\frac{-x+1+2}{2-x}\right]-\left[\frac{(x+1)(2-x)+4}{(2+x)(2-x)}\right]$$

$$=\frac{3-x}{2-x}-\left[\frac{2x-x^2+2-x+4}{(2+x)(2-x)}\right]$$

$$=\frac{3-x}{2-x}-\left[\frac{6+x-x^2}{(2+x)(2-x)}\right]$$

$$=\frac{3-x}{2-x}-\left[\frac{6+3x-2x-x^2}{(2+x)(2-x)}\right]$$

$$=\frac{3-x}{2-x}-\left[\frac{3(2+x)-x(2+x)}{(2+x)(2-x)}\right]$$

$$=\frac{3-x}{2-x}-\left[\frac{\cancel{(2+x)}(3-x)}{\cancel{(2+x)}(2-x)}\right]$$

$$=\frac{3-x}{2-x}-\frac{3-x}{2-x}$$

$$=\frac{3-x-3+x}{2-x}$$

$$=\frac{0}{2-x}$$

$$=0$$

Q8. What rational expression should be subtracted from $\frac{2x^2+2x-7}{x^2+x-6}$ **to get** $\frac{x-1}{x-2}=?$

Sol: Let the required expression be A,

then $\frac{2x^2+2x-7}{x^2+x-6}-A=\frac{x-1}{x-2}$

or $\frac{2x^2+2x-7}{x^2+x-6}-\frac{x-1}{x-2}=A$

So $A=\frac{2x^2+2x-7}{x^2+3x-2x-6}-\frac{x-1}{x-2}$

$=\frac{2x^2+2x-7}{x(x+3)-2(x+3)}-\frac{x-1}{x-2}$

$=\frac{2x^2+2x-7}{(x+3)(x-2)}-\frac{x-1}{x-2}$

$=\frac{2x^2+2x-7-(x-1)(x+3)}{(x+3)(x-2)}$

$=\frac{2x^2+2x-7-(x^2-x+3x-3)}{(x+3)(x-2)}$

$=\frac{(2x^2+2x-7)-(x^2+2x-3)}{(x+3)(x-2)}$

$=\frac{2x^2+2x-7-x^2-2x+3}{(x+3)(x-2)}$

$=\frac{x^2-4}{(x+3)(x-2)}$

$=\frac{(x)^2-(2)^2}{(x+3)(x-2)}$

$=\frac{(x+2)\cancel{(x-2)}}{(x+3)\cancel{(x-2)}}$

$=\frac{x+2}{x+3}$

Perform the indicated operations and simplify to the lowest forms.

Q9. $\frac{x^2+x-6}{x^2-x-6}\times\frac{x^2-4}{x^2-9}$

$=\frac{x^2+3x-2x-6}{x^2-3x+2x-6}\times\frac{(x)^2-(2)^2}{(x)^2-(3)^2}$

$=\frac{x(x+3)-2(x+3)}{x(x-3)+2(x-3)}\times\frac{(x+2)(x-2)}{(x+3)(x-3)}$

$=\frac{\cancel{(x+3)}(x-2)}{(x-3)\cancel{(x+2)}}\times\frac{\cancel{(x+2)}(x-2)}{\cancel{(x+3)}(x-3)}$

$=\frac{(x-2)^2}{(x-3)^2}$

Q10. $\frac{x^3-8}{x^2-4}\times\frac{x^2+6x+8}{x^2-2x+1}$

$=\frac{(x)^3-(2)^3}{(x)^2-(2)^2}\times\frac{x^2+2x+4x+8}{x^2-x-x+1}$

$=\frac{\cancel{(x-2)}\left[(x)^2+(x)(2)+(2)^2\right]}{\cancel{(x-2)}(x+2)}\times\frac{x(x+2)+4(x+2)}{x(x-1)-1(x-1)}$

$=\frac{x^2+2x+4}{\cancel{x+2}}\times\frac{\cancel{(x+2)}(x+4)}{(x-1)(x-1)}$

$=\frac{(x^2+2x+4)(x+4)}{(x-1)^2}$

Q11. $\frac{x^4-8x}{2x^2+5x-3}\times\frac{2x-1}{x^2+2x+4}\times\frac{x+3}{x^2-2x}$

$=\frac{x(x^3-8)}{2x^2+6x-x-3}\times\frac{2x-1}{x^2+2x+4}\times\frac{x+3}{x(x-2)}$

$=\frac{x\left[(x)^3-(2)^3\right]}{2x(x+3)-1(x+3)}\times\frac{2x-1}{x^2+2x+4}\times\frac{x+3}{x(x-2)}$

$=\frac{\cancel{x}\cancel{(x-2)}\cancel{(x^2+2x+4)}}{\cancel{(x+3)}\cancel{(2x-1)}}\times\frac{\cancel{2x-1}}{\cancel{x^2+2x+4}}\times\frac{\cancel{x+3}}{\cancel{x}\cancel{(x-2)}}$

$=1$

Q12. $\frac{2y^2+7y-4}{3y^2-13y+4}\div\frac{4y^2-1}{6y^2+y-1}$

$$=\frac{2y^2+8y-y-4}{3y^2-y-12y+4}\div\frac{(2y)^2-(1)^2}{6y^2+3y-2y-1}$$

$$=\frac{2y(y+4)-1(y+4)}{y(3y-1)-4(3y-1)}\div\frac{(2y+1)(2y-1)}{3y(2y+1)-1(2y+1)}$$

$$=\frac{(y+4)\cancel{(2y-1)}}{\cancel{(3y-1)}(y-4)}\div\frac{\cancel{(2y+1)}(2y-1)}{\cancel{(2y+1)}\cancel{(3y-1)}}$$

$$=\frac{(y+4)\cancel{(2y-1)}}{\cancel{(3y-1)}(y-4)}\times\frac{\cancel{(2y+1)}(3y-1)}{\cancel{(2y+1)}\cancel{(2y-1)}}$$

$$=\frac{y+4}{y-4}$$

Q13. $\left[\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right]\div\left[\frac{x+y}{x-y}-\frac{x-y}{x+y}\right]$

$$=\left[\frac{(x^2+y^2)^2-(x^2-y^2)^2}{(x^2-y^2)(x^2+y^2)}\right]\div\left[\frac{(x+y)^2-(x-y)^2}{(x-y)(x+y)}\right]$$

$$=\frac{x^4+y^4+2x^2y^2-(x^4+y^4-2x^2y^2)^2}{(x^2-y^2)(x^2+y^2)}$$

$$\div\frac{x^2+y^2+2xy-x^2-y^2+2xy}{x^2-y^2}$$

$$=\frac{\cancel{x^4}+\cancel{y^4}+2x^2y^2-\cancel{x^4}-\cancel{y^4}+2x^2y^2}{(x^2-y^2)(x^2+y^2)}$$

$$\div\frac{\cancel{x^2}+\cancel{y^2}+2xy-\cancel{x^2}-\cancel{y^2}+2xy}{x^2-y^2}$$

$$=\frac{4x^2y^2}{(x^2-y^2)(x^2+y^2)}\div\frac{4xy}{x^2-y^2}$$

$$=\frac{\cancel{4}\cancel{x^2}\cancel{y^2}}{\cancel{(x^2-y^2)}(x^2+y^2)}\times\frac{\cancel{x^2-y^2}}{\cancel{4xy}}$$

$$=\frac{xy}{x^2+y^2}$$

Square Root of Algebraic Expression

The square root of a given expression $p(x)$ as another expression $q(x)$ such that $q(x).q(x) = p(x)$.

As $5\times5=25$, so square root of 25 is 5

It means we can find square root of the expression $p(x)$ if it can be expressed as a perfect square.

Example

Use factorization to find the square root of the expression

$$4x^2-12x+9$$

Solution

We have, $4x^2-12x+9$

$$=4x^2-6x-6x+9=2x(2x-3)-3(2x-3)$$

$$=\quad(2x-3)(2x-3)=(2x-3)^2$$

Hence $\sqrt{4x^2-12x+9}$

$$=\quad\pm(2x-3)$$

Example

Find the square root of

$$x^2+\frac{1}{x^2}+12\left(x+\frac{1}{x}\right)+38,x\neq0$$

Solution

We have $x^2+\frac{1}{x^2}+12\left(x+\frac{1}{x}\right)+38$

$$=\quad x^2+\frac{1}{x^2}+2+12\left(x+\frac{1}{x}\right)+36,$$

(adding and subtracting 2)

$$= \left(x+\frac{1}{x}\right)^2 + 2\left(x+\frac{1}{x}\right)(6)+(6)^2$$

$$= \left[\pm\left(x+\frac{1}{x}+6\right)\right]^2;$$

since $a^2+2ab+b^2=(a+b)^2$

Hence the required square root is $\pm\left(x+\frac{1}{x}+6\right)$

Example

Find the square root of $4x^4+12x^3+x^2-12x+4$

Solution

$$\begin{array}{r|l}
 & 2x^2+3x-2 \\
\hline
2x^2 & \cancel{4x^4}+12x^3+x^2-12x+4 \\
 & \cancel{4x^4} \\
\hline
4x^2+3x & \cancel{12x^3}+x^2-12x+4 \\
 & \cancel{12x^3} \pm 9x^2 \\
\hline
4x^2+6x-2 & -8x^2-12x+4 \\
 & \mp 8x^2 \mp 12x \pm 4 \\
\hline
 & 0
\end{array}$$

Thus square root of given expression is $\pm(2x^2+3x-2)$

Example 2

Find the square root of the expression

$$4\frac{x^2}{y^2}+8\frac{x}{y}+16+12\frac{y}{x}+9\frac{y^2}{x^2}$$

Solution

We note that the given expression is in descending powers of x.

$$\begin{array}{r|l}
 & 2\frac{x}{y}+2+3\frac{y}{x} \\
\hline
2\frac{x}{y} & 4\frac{x^2}{y^2}+8\frac{x}{y}+16+12\frac{y}{x}+9\frac{y^2}{x^2} \\
 & \cancel{\pm 4\frac{x^2}{y^2}} \\
\hline
4\frac{x}{y}+2 & 8\frac{x}{y}+16 \\
 & \pm 8\frac{x}{y} \pm 4 \\
\hline
4\frac{x}{y}+4+3\frac{y}{x} & 12+12\frac{y}{x}+9\frac{y^2}{x^2} \\
 & \pm 12 \pm 12\frac{y}{x} \pm 9\frac{y^2}{x^2} \\
\hline
 & 0
\end{array}$$

Hence the square root of given expression is $\pm\left(2\frac{x}{y}+2+3\frac{y}{x}\right)$

Example

To make the expression $x^4-10x^3+33x^2-42x+20$ a perfect square,

(i) What should be added to it?

(ii) What should be subtracted from it?

(iii) What should be the value of x?

$$\begin{array}{r|l}
 & x^2-5x+4 \\
\hline
x^2 & x^4-10x^3+33x^2-42x+20 \\
 & \pm x^4 \\
\hline
2x^2-5x & -10x^3+33x^2 \\
 & \underset{+}{-}10x^3 \underset{-}{+} 25x^2 \\
\hline
2x^2-10x+4 & 8x^2-42x+20 \\
 & \underset{+}{-}8x^2 \underset{-}{-}40x+16 \\
\hline
 & -2x+4
\end{array}$$

For making the given expression a perfect square the remainder must be zero.

Hence

(i) We should add $(2x-4)$ to the given expression

(ii) We should subtract $(-2x+4)$ from the given expression

(iii) We should take $-2x+4=0$ to find the value of x. This gives the required value of x i.e., $x=2$.

Exercise 6.3

Q1. Use factorization to find the square root of the following expressions.

i) $4x^2-12xy+9y^2$

$=(2x)^2-2(2x)(3y)+(3y)^2$

$=(2x-3y)^2$

Hence $\sqrt{4x^2-12xy+9y^2}$

$=\sqrt{(2x-3y)^2}$

$=\pm(2x-3y)$

ii) $x^2-1+\dfrac{1}{4x^2}$

$=(x)^2-2(x)\left(\dfrac{1}{2x}\right)+\left(\dfrac{1}{2x}\right)^2$

Hence $\sqrt{x^2-1+\dfrac{1}{4x^2}}$

$=\sqrt{\left(x-\dfrac{1}{2x}\right)^2}$

$=\pm\left(x-\dfrac{1}{2x}\right)$

iii) $\dfrac{1}{16}x^2-\dfrac{1}{12}xy+\dfrac{1}{36}y^2$

$=\left(\dfrac{1}{4}x\right)^2-2\left(\dfrac{1}{4}x\right)\left(\dfrac{1}{6}y\right)+\left(\dfrac{1}{6}y\right)^2$

$=\left(\dfrac{1}{4}x-\dfrac{1}{6}y\right)^2$

Hence $\sqrt{\dfrac{1}{16}x^2-\dfrac{1}{12}xy+\dfrac{1}{36}y^2}$

$=\sqrt{\left(\dfrac{1}{4}x-\dfrac{1}{6}y\right)^2}$

$=\pm\left(\dfrac{1}{4}x-\dfrac{1}{6}y\right)$

iv) $4(a+b)^2-12\left(a^2-b^2\right)+9(a-b)^2$

$=[2(a+b)]^2-2\times2(a+b)\times3(a-b)+[3(a-b)]^2$

$=[2(a+b)-3(a-b)]^2$

$=(-a+5b)^2$

$=(5b-a)^2$

Hence $\sqrt{4(a+b)^2-12\left(a^2-b^2\right)+9(a-b)^2}$

$=\sqrt{(5b-a)^2}$

$=\pm(5b-a)$

v) $\dfrac{4x^6-12x^3y^3+9y^6}{9x^4+24x^2y^2+16y^4}$

$=\dfrac{\left(2x^3\right)^2-2\left(2x^3\right)\left(3y^3\right)+\left(3y^3\right)^2}{\left(3x^2\right)^2+2\left(3x^2\right)\left(4y^2\right)+\left(4y^2\right)^2}$

$$=\frac{\left(2x^3-3y^3\right)^2}{\left(3x^2+4y^2\right)^2}$$

Hence $\sqrt{\frac{4x^6-12x^3y^3+9y^6}{9x^4+24x^2y^2+16y^4}}$

$$=\sqrt{\left(\frac{2x^3-3y^3}{3x^2+4y^2}\right)^2}$$

$$=\pm\left(\frac{2x^3-3y^3}{3x^2+4y^2}\right)$$

vi) $\left(x+\frac{1}{x}\right)^2-4\left(x-\frac{1}{x}\right) \qquad (x\neq 0)$

$$=(x)^2+\left(\frac{1}{x}\right)^2+2(\cancel{x})\left(\frac{1}{\cancel{x}}\right)-4\left(x-\frac{1}{x}\right)$$

$$=x^2+\frac{1}{x^2}+2-4\left(x-\frac{1}{x}\right) \ldots\ldots\text{(i)}$$

Let $x-\frac{1}{x}=a$

Squaring $\left(x-\frac{1}{x}\right)^2=(a)^2$

$$x^2+\frac{1}{x^2}-2=a^2$$

$$x^2+\frac{1}{x^2}=a^2+2$$

So expression (i) becomes

$$=a^2+2+2-4a$$
$$=a^2-4a+4$$
$$=(a)^2-2(a)(2)+(2)^2$$
$$=(a-2)^2$$

Putting value of 'a'

$$=\left(x-\frac{1}{x}-2\right)^2$$

Hence $=\sqrt{\left(x-\frac{1}{x}-2\right)^2}$

$$=\pm\left(x-\frac{1}{x}-2\right)$$

vii) $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12 \ldots\text{(i)}$

Let $x+\frac{1}{x}=a$

Squaring $\left(x+\frac{1}{x}\right)^2=(a)^2$

$$x^2+\frac{1}{x^2}+2=a^2$$

$$x^2+\frac{1}{x^2}=a^2-2$$

So expression (i) becomes

$$=\left(a^2-2\right)^2-4(a)^2+12$$
$$=\left(a^2\right)^2-2\left(a^2\right)(2)+(2)^2-4a^2+12$$
$$=a^4-4a^2+4-4a^2+12$$
$$=a^4-8a^2+16$$
$$=\left(a^2\right)^2-2\left(a^2\right)(4)+(4)^2$$
$$=\left(a^2-4\right)^2$$

Putting values of a^2

$$=\left(x^2+\frac{1}{x^2}+2-4\right)^2$$

$$=\left(x^2+\frac{1}{x^2}-2\right)^2$$

Hence $=\sqrt{\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12}$

$$=\sqrt{\left(x^2+\frac{1}{x^2}-2\right)^2}$$

$$=\pm\left(x^2+\frac{1}{x^2}-2\right)$$

viii) $(x^2+3x+2)(x^2+4x+3)(x^2+5x+6)$

$=(x^2+x+2x+2)(x^2+x+3x+3)(x^2+2x+3x+6)$

$=[x(x+1)+2(x+1)][x(x+1)+3(x+1)][x(x+2)+3(x+2)]$

$=(x+1)(x+2)(x+1)(x+3)(x+2)(x+3)$

$=(x+1)^2(x+2)^2(x+3)^2$

Hence

$$\sqrt{(x^2+3x+2)(x^2+4x+3)(x^2+5x+6)}$$

$$=\sqrt{(x+1)^2(x+2)^2(x+3)^2}$$

$$=\pm(x+1)(x+2)(x+3)$$

ix) $(x^2+8x+7)(2x^2-x-3)(2x^2+11x-21)$

$=(x^2+x+7x+7)(2x^2+2x-3x-3)(2x^2+14x-3x-21)$

$=[x(x+1)+7(x+1)][2x(x+1)-3(x+1)]$
$[2x(x+7)-3(x+7)]$

$=(x+1)(x+7)(x+1)(2x-3)(x+7)(2x-3)$

$=(x+1)^2(x+7)^2(2x-3)^2$

Hence

$$\sqrt{(x^2+8x+7)(2x^2-x-3)(2x^2+11x-21)}$$

$$=\sqrt{(x+1)^2(x+7)^2(2x-3)^2}$$

$$=\pm(x+1)(x+7)(2x-3)$$

Q2. Use division method to find the square root of the following expressions.

i) $4x^2+12xy+9y^2+16x+24y+16$

$$\begin{array}{r|l}
 & 2x+3y+4 \\
\hline
2x & 4x^2+12xy+9y^2+16x+24y+16 \\
 & \underline{4x^2} \\
\hline
4x+3y & 12xy+9y^2+16x+24y+16 \\
 & \pm 12xy \pm 9y^2 \\
\hline
4x+6y+4 & 16x+24y+16 \\
 & \pm 16x \pm 24y \pm 16 \\
\hline
 & 0
\end{array}$$

Hence the square root of given expression is

$\pm(2x+3y+4)$

ii) $x^4-10x^3+37x^2-60x+36$

$$\begin{array}{r|l}
 & x^2-5x+6 \\
\hline
x^2 & x^4-10x^3+37x^2-60x+36 \\
 & -x^4 \\
\hline
2x^2-5x & -10x^3+37x^2-60x+36 \\
 & \mp 10x^3 \pm 25x^2 \\
\hline
2x^2-10x+6 & 12x^2-60x+36 \\
 & \pm 12x^2 \mp 60x \pm 36 \\
\hline
 & 0
\end{array}$$

Hence $\sqrt{x^4-10x^3+37x^2-60x+36}$

$=\pm(x^2-5x+6)$

iii) $9x^4-6x^3+7x^2-2x+1$

$$
\begin{array}{r|l}
 & 3x^2-x+1 \\
\hline
3x^2 & 9x^4-6x^3+7x^2-2x+1 \\
 & -9x^4 \\
\hline
6x^2-x & \cancel{-6x^3}+7x^2-2x+1 \\
 & \cancel{\mp 6x^3}\pm x^2 \\
\hline
6x^2-2x+1 & \cancel{6x^2}-\cancel{2x}+\cancel{1} \\
 & \cancel{-6x^2}\mp\cancel{2x}\pm\cancel{1} \\
\hline
 & 0
\end{array}
$$

Hence $\sqrt{9x^4-6x^3+7x^2-2x+1}$

$=\pm\left(3x^2-x+1\right)$

iv) $4+25x^2-12x-24x^3+16x^4$

In descending order

$=16x^4-24x^3+25x^2-12x+4$

$$
\begin{array}{r|l}
 & 4x^2-3x+2 \\
\hline
4x^2 & 16x^4-24x^3+25x^2-12x+4 \\
 & -16x^4 \\
\hline
8x^2-3x & \cancel{-24x^3}+25x^2-12x+4 \\
 & \cancel{\mp 24x^3}\pm 9x^2 \\
\hline
8x^2-6x+2 & \cancel{16x^2}-\cancel{12x}+\cancel{4} \\
 & \cancel{-16x^2}\mp\cancel{12x}\pm\cancel{4} \\
\hline
 & 0
\end{array}
$$

Hence $\sqrt{16x^4-24x^3+25x^2-12x+4}$

$=\pm\left(4x^2-3x+2\right)$

v) $\dfrac{x^2}{y^2}-10\dfrac{x}{y}+27-10\dfrac{y}{x}+\dfrac{y^2}{x^2}$

$(x\neq 0,\ y\neq 0)$

Hence

$$
\begin{array}{r|l}
 & \frac{x}{y}-5+\frac{y}{x} \\
\hline
\frac{x}{y} & \frac{x^2}{y^2}-10\frac{x}{y}+27-10\frac{y}{x}+\frac{y^2}{x^2} \\
 & -\frac{x^2}{y^2} \\
\hline
\frac{2x}{y}-5 & \cancel{-10\frac{x}{y}}+27-10\frac{y}{x}+\frac{y^2}{x^2} \\
 & \cancel{\mp 10\frac{x}{y}}\pm 25 \\
\hline
\frac{2x}{y}-10+\frac{y}{x} & \cancel{2}-\cancel{10\frac{y}{x}}+\cancel{\frac{y^2}{x^2}} \\
 & \cancel{2}\mp\cancel{10\frac{y}{x}}\pm\cancel{\frac{y^2}{x^2}} \\
\hline
 & 0
\end{array}
$$

$\sqrt{\dfrac{x^2}{y^2}-10\dfrac{x}{y}+27-10\dfrac{y}{x}+\dfrac{y^2}{x^2}}$

The required square root

$=\pm\left(\dfrac{x}{y}-5+\dfrac{y}{x}\right)$

Q3. Find the value of 'k' for which the following expression will become a perfect square?

i) $4x^4-12x^3+37x^2-42x+k$

$$
\begin{array}{r|l}
 & 2x^2-3x+7 \\
\hline
2x^2 & 4x^4-12x^3+37x^2-42x+k \\
 & -4x^4 \\
\hline
4x^2-3x & \cancel{-12x^3}+37x^2-42x+k \\
 & \cancel{\mp 12x^3}\pm 9x^2 \\
\hline
4x^2-6x+7 & \cancel{28x^2}-\cancel{42x}+k \\
 & \cancel{-28x^2}\mp\cancel{42x}\pm 49 \\
\hline
 & k-49
\end{array}
$$

As given that the given expression is a perfect square, so

Remainder $=0$

$k-49=0$

$$\boxed{k=49}$$

ii) $x^4-4x^3+10x^2-kx+9$

$$
\begin{array}{r|l}
 & x^2-2x+3 \\
\hline
x^2 & x^4-4x^3+10x^2-kx+9 \\
 & -x^4 \\
\hline
2x^2-2x & -\cancel{4x^3}+10x^2-kx+9 \\
 & \mp\cancel{4x^3}\pm 4x^2 \\
\hline
2x^2-4x+3 & \cancel{6x^2}-kx+\cancel{9} \\
 & -\cancel{6x^2}\mp 12x\pm\cancel{9} \\
\hline
 & (-k+12)x
\end{array}
$$

As given that the given expression is a perfect square, so

$$\text{Remainder}=0$$

$$(-k+12)x=0$$

As $x\neq 0$, so $-k+12=0$

$$\Rightarrow \boxed{k=12}$$

Q4. Find the values of 'l' and 'm' for which the following expression will become perfect square.

i) $x^4+4x^3+16x^2+lx+m$

$$
\begin{array}{r|l}
 & x^2+2x+6 \\
\hline
x^2 & \cancel{x^4}+4x^3+16x^2+lx+m \\
 & -\cancel{x^4} \\
\hline
2x^2+2x & \cancel{4x^3}+16x^2+lx+m \\
 & -\cancel{4x^3}\pm 4x^2 \\
\hline
2x^2+4x+6 & \cancel{12x^2}+lx+m \\
 & -\cancel{12x^2}\pm 24x\pm 36 \\
\hline
 & (l-24)x+(m-36)
\end{array}
$$

As the given expression is to be a perfect square, so

$$\text{Remainder}=0$$

$$(l-24)x+(m-36)=0$$

As $x\neq 0$, so $l-24=0$ and $m-36=0$

$\Rightarrow \boxed{l=24}$ and $\boxed{m=36}$

ii) $49x^4-70x^3+109x^2+lx-m$

$$
\begin{array}{r|l}
 & 7x^2-5x+6 \\
\hline
7x^2 & 49x^4-70x^3+109x^2+lx-m \\
 & -49x^4 \\
\hline
14x^2-5x & -\cancel{70x^3}+109x^2+lx-m \\
 & \mp\cancel{70x^3}\pm 25x^2 \\
\hline
14x^2-10x+6 & \cancel{84x^2}+lx-m \\
 & -\cancel{84x^2}\mp 60x\pm 36 \\
\hline
 & (l+60)x-m-36
\end{array}
$$

As the given expression is to be a perfect square, so

$$(l+60)x-m-36=0$$

As $x\neq 0$, so $l+60=0$ and $-m-36=0$

$\Rightarrow \boxed{l=-60}$ and $\boxed{m=-36}$

Q5. To make the expression $9x^4-12x^3+22x^2-13x+12$ a perfect square.

i) What should be added to it?

ii) What should be subtracted from it?

iii) What should be the value of 'x'?

$$
\begin{array}{r|l}
 & 3x^2-2x+3 \\
\hline
3x^2 & 9x^4-12x^3+22x^2-13x+12 \\
 & -9x^4 \\
\hline
6x^2-2x & -\cancel{12x^3}+22x^2-13x+12 \\
 & \mp\cancel{12x^3}\pm 4x^2 \\
\hline
6x^2-4x+3 & \cancel{18x^2}-13x+12 \\
 & -\cancel{18x^2}\mp 12x\pm 9 \\
\hline
 & -x+3
\end{array}
$$

To make the given expression a complete square

i) $x-3$ should be added

ii) $-x+3$ should be subtracted

iii) For value of 'x'

Remainder = 0

$-x+3=0$

$\boxed{x=3}$

Q6. Find H.C.F of following by factorization

$8x^4-128,\ 12x^3-96.$

Solution:

$$\begin{aligned}8x^4-128 &= 8(x^4-16)\\ &= 8((x^2)^2-(4)^2)\\ &= 8(x^2+4)(x^2-4)\\ &= 8(x^2+4)(x+2)(x-2)\end{aligned}$$

$$\begin{aligned}12x^3-96 &= 12(x^3-8)\\ &= 12(x^3-2^3)\\ &= 12(x-2)(x^2+2x+4)\end{aligned}$$

Common factor $= 4(x-2)$

H.C.F $= 4(x-2)$

Q7. Find H.C.F of following by division method.

$y^3+3y^2-3y-9,\ y^3+3y^2-8y-24$

Solution:

$$\begin{array}{r|l} & 1 \\ y^3+3y^2-3y-9 & y^3+3y^2-8y-24 \\ & -y^3 \pm 3y^2 \mp 3y \mp 9 \\ & -5y-15 \\ & -5(y+3) \end{array}$$

$$\begin{array}{r|l} & y^2-3 \\ (y+3) & y^3+3y^2-3y-9 \\ & -y^3 \pm 3y^2 \\ & -3y-9 \\ & \mp 3y \pm 9 \\ & \times \end{array}$$

H.C.F = y + 3

Q8. Find L.C.M of following by factorization.

$12x^2\ 75,\ 6x^2-13x-5,\ 4x^2-20x+25$

Solution:

$$\begin{aligned}12x^2-75 &= 3(4x^2-25)\\ &= 3((2x)^2-(5)^2)\\ &= 3(2x+5)(2x-5)\end{aligned}$$

$$\begin{aligned}6x^2-13x-5 &= 6x^2-15x+2x-5\\ &= 3x(2x-5)+1(2x-5)\\ &= (3x+1)(2x-5)\end{aligned}$$

$$\begin{aligned}4x^2-20x+25 &= (2x)^2+(5)^2-2(2x)(5)\\ &= (2x-5)^2\\ &= (2x-5)(2x-5)\end{aligned}$$

$$\begin{aligned}\text{L.C.M} &= (2x-5)^2 \times 3(2x+5)(3x+1)\\ &= 3(2x-5)^2(2x+5)(3x+1)\end{aligned}$$

Q9. If H.C.F of $x^4+3x^3+5x^2+26x+56$ and $x^4+2x^3-4x^2-x+28$ is x^2+5x+7, find the

Solution:

$$\text{L.C.M} = \frac{(x^4+3x^3+5x^2+26x+56)(x^4+2x^3-4x^2-x+28)}{x^2+5x+7}$$

$$\begin{array}{r|l} & x^2-2x+8 \\ x^2+5x+7 & \cancel{x^4}+3x^3+5x^2+26x+56 \\ & \cancel{-x^4} \pm 5x^3 \pm 7x^2 \\ & -\cancel{2x^3}-2x^2+26x+56 \\ & \mp \cancel{2x^3} \mp 10x^2 \mp 14x \\ & 8x^2+40x+56 \\ & -8x^2 \pm 40x \pm 56 \\ & \times \end{array}$$

L.C.M

$=(x^2-2x+8)(x^4+2x^3-4x^2-x+28)$

Q10. Simplify

(i) $$\frac{3}{x^3+x^2+x+1}-\frac{3}{x^3-x^2+x-1}$$

$$\frac{3}{(x^2+1)(x+1)}-\frac{3}{(x^2+1)(x-1)}$$

$$=\frac{3(x-1)-3(x+1)}{(x^2+1)(x+1)(x-1)}$$

$$=\frac{\cancel{3x}-3-\cancel{3x}-3}{(x^2+1)(x+1)(x-1)}$$

$$=\frac{-6}{(x^2+1)(x+1)(x-1)}$$

$$=\frac{-6}{(x^2+1)(x^2-1)}$$

ختم نبوت ﷺ زندہ باد — عظمت صحابہ زندہ باد

السلام علیکم ورحمۃ اللہ وبرکاتہ:

معزز ممبران: آپ کا وٹس ایپ گروپ ایڈمن "**اردو بکس**" آپ سے مخاطب ہے۔

آپ تمام ممبران سے گزارش ہے کہ:

- گروپ میں صرف PDF کتب پوسٹ کی جاتی ہیں لہذا کتب کے متعلق اپنے کمنٹس / ریویوز ضرور دیں۔ گروپ میں بغیر ایڈمن کی اجازت کے کسی بھی قسم کی (اسلامی و غیر اسلامی، اخلاقی، تحریری) پوسٹ کرنا سختی سے منع ہے۔
- گروپ میں معزز، پڑھے لکھے، سلجھے ہوئے ممبرز موجود ہیں اخلاقیات کی پابندی کریں اور گروپ رولز کو فالو کریں بصورت دیگر معزز ممبرز کی بہتری کی خاطر ریموو کر دیا جائے گا۔
- کوئی بھی ممبر کسی بھی ممبر کو انباکس میں میسیج، مس کال، کال نہیں کرے گا۔ رپورٹ پر فوری ریموو کر کے کاروائی عمل میں لائے جائے گی۔
- ہمارے کسی بھی گروپ میں سیاسی و فرقہ واریت کی بحث کی قطعاً کوئی گنجائش نہیں ہے۔
- اگر کسی کو بھی گروپ کے متعلق کسی قسم کی شکایت یا تجویز کی صورت میں ایڈمن سے رابطہ کیجئے۔
- سب سے اہم بات:

گروپ میں کسی بھی قادیانی، مرزائی، احمدی، گستاخِ رسول، گستاخِ امہات المؤمنین، گستاخِ صحابہ و خلفائے راشدین حضرت ابو بکر صدیق، حضرت عمر فاروق، حضرت عثمان غنی، حضرت علی المرتضیٰ، حضرت حسنین کریمین رضوان اللہ تعالیٰ اجمعین، گستاخ اہلبیت یا ایسے غیر مسلم جو اسلام اور پاکستان کے خلاف پراپیگنڈا میں مصروف ہیں یا ان کے روحانی و ذہنی سپورٹرز کے لئے کوئی گنجائش نہیں ہے لہذا ایسے اشخاص بالکل بھی گروپ جوائن کرنے کی زحمت نہ کریں۔ معلوم ہونے پر فوراً ریموو کر دیا جائے گا۔

- تمام کتب انٹرنیٹ سے تلاش / ڈاؤنلوڈ کر کے **فری آف کاسٹ** وٹس ایپ گروپ میں شیئر کی جاتی ہیں۔ جو کتاب نہیں ملتی اس کے لئے معذرت کر لی جاتی ہے۔ جس میں محنت بھی صَرف ہوتی ہے لیکن ہمیں آپ سے صرف دعاؤں کی درخواست ہے۔
- عمران سیریز کے شوقین کیلئے علیحدہ سے عمران سیریز گروپ موجود ہے۔
- **لیڈیز کے لئے الگ گروپ کی سہولت موجود ہے جس کے لئے ویریفیکیشن ضروری ہے۔**
- اردو کتب / عمران سیریز یا سٹڈی گروپ میں ایڈ ہونے کے لئے ایڈمن سے وٹس ایپ پر بذریعہ میسیج رابطہ کریں اور جواب کا انتظار فرمائیں۔ برائے مہربانی اخلاقیات کا خیال رکھتے ہوئے موبائل پر کال یا ایم ایس کرنے کی کوشش ہرگز نہ کریں۔ ورنہ گروپس سے تو ریموو کیا ہی جائے گا بلاک بھی کیا جائے گا۔

نوٹ: ہمارے کسی گروپ کی کوئی فیس نہیں ہے۔ سب فی سبیل اللہ ہے

0306-7163117 — محمد سلمان سلیم

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0333-8033313 — راؤ ایاز

پاکستان زندہ باد — پاکستان پائندہ باد

اللہ تبارک تعالیٰ ہم سب کا حامی و ناصر ہو

$$= \frac{-6}{x^4-1} = \frac{6}{1-x^4} \text{ Ans.}$$

(ii) $\dfrac{a+b}{a^2-b^2} \div \dfrac{a^2-ab}{a^2-2ab+b^2}$

$$= \frac{a+b}{(a-b)(a+b)} \div \frac{a(a-b)}{(a-b)^2}$$

$$= \frac{1}{a-b} \div \frac{a}{a-b}$$

$$= \frac{1}{\cancel{a-b}} \times \frac{\cancel{a-b}}{a}$$

$$= \frac{1}{a}$$

Q11. Find square root by using factorization

$\left(x^2+\dfrac{1}{x^2}\right)+10\left(x+\dfrac{1}{x}\right)+27 \qquad (x \neq 0)$

Solution:

$$= \left(x^2+\frac{1}{x^2}\right)+10\left(x+\frac{1}{x}\right)+25+2$$

$$= x^2+\frac{1}{x^2}+2+10\left(x+\frac{1}{2}\right)+25$$

$$= \left(x+\frac{1}{x}\right)^2+10\left(x+\frac{1}{x}\right)+25$$

Let $x+\dfrac{1}{x}=a$

$$= a^2+10a+25$$

$$= (a+5)^2$$

Taking square root

$$= \sqrt{\left[\pm(a+5)\right]^2}$$

$$= \pm(a+5)$$

$$= \pm\left(x+\frac{1}{x}+5\right)$$

Q12. Find square root by using division method.

$\dfrac{4x^2}{y^2}+\dfrac{20x}{y}+13-\dfrac{30y}{x}+\dfrac{9y^2}{x^2} \quad (x, y \neq 0)$

Solution:

$$
\begin{array}{r|l}
 & \dfrac{2x}{y}+5-\dfrac{3y}{x} \\
\hline
\dfrac{2x}{y} & \dfrac{4x^2}{y^2}+\dfrac{20x}{y}+13-\dfrac{30y}{x}+\dfrac{9y^2}{x^2} \\
 & -\dfrac{4x^2}{y^2} \\
\hline
\dfrac{4x}{y}+5 & \dfrac{20}{y}x+13 \\
 & -\dfrac{20}{y}x \pm 25 \\
\hline
\dfrac{4x}{y}+10-\dfrac{3y}{x} & -12-\dfrac{30y}{x}+\dfrac{9y^2}{x^2} \\
 & \mp 12 \mp \dfrac{30y}{x} \pm \dfrac{9y^2}{x^2} \\
\hline
 & \times
\end{array}
$$

Required square root $= \pm\left(\dfrac{2x}{y}+5-\dfrac{3y}{x}\right)$

Objective

1. H.C.F of p^3q-pq^3 and $p^5q^2-p^2q^5$ is ___
(a) $pq(p^2-q^2)$ (b) $pq(p-q)$
(c) $p^2q^2(p-q)$ (d) $pq(p^3-q^3)$

2. H.C.F. of $5x^2y^2$ and $20x^3y^3$ is:___
(a) $5x^2y^2$ (b) $20x^3y^3$
(c) $100x^5y^5$ (d) $5xy$

3. H.C.F of $x-2$ and x^2+x-6 is ___
(a) x^2+x-6 (b) $x+2$
(c) $x-2$ (d) $x+2$

4. H.C.F of a^3+b^3 and a^2-ab+b^2 is ___
(a) $a+b$
(b) a^2-ab+b^2
(c) $(a-b)^2$ (d) a^2+b^2

5. H.C.F of x^2-5x+6 and x^2-x-6 is __:
(a) $x-3$ (b) $x+2$
(c) x^2-4 (d) $x-2$

6. H.C.F of a^2-b^2 and a^3-b^3 is___
(a) $a-b$ (b) $a+b$
(c) a^2+ab+b^2 (d) a^2-ab+b^2

7. H.C.F of x^2+3x+2, x^2+4x+3, x^2+5x+4 is:
(a) $x+1$ (b) $(x+1)(x+2)$
(c) $(x+3)$ (d) $(x+4)(x+1)$

8. L.C.M of $15x^2$, $45xy$ and $30xyz$ is__
(a) $90xyz$ (b) $90x^2yz$
(c) $15xyz$ (d) $15x^2yz$

9. L.C.M of a^2+b^2 and a^4-b^4 is:__
(a) a^2+b^2 (b) a^2-b^2
(c) a^4-b^4 (d) $a-b$

10. The product of two algebraic expression is equal to the ___ of their H.C.F and L.C.M.
(a) Sum
(b) Difference
(c) Product
(d) Quotient

11. Simplify $\dfrac{a}{9a^2-b^2}+\dfrac{1}{3a-b}=$ ___
(a) $\dfrac{4a}{9a^2-b^2}$
(b) $\dfrac{4a-b}{9a^2-b^2}$
(c) $\dfrac{4a+b}{9a^2-b^2}$
(d) $\dfrac{b}{9a^2-b^2}$

12. Simplify $\dfrac{a^2+5a-14}{a^2-3a-18}\times\dfrac{a+3}{a-2}=$ ___
(a) $\dfrac{a+7}{a-6}$ (b) $\dfrac{a+7}{a-2}$
(c) $\dfrac{a+3}{a-6}$ (d) $\dfrac{a-3}{a+2}$

13. Simplify
$\dfrac{a^3-b^3}{a^4-b^4}\div\left(\dfrac{a^2+ab+b^2}{a^2+b^2}\right)=$___
(a) $\dfrac{1}{a+b}$ (b) $\dfrac{1}{a-b}$
(c) $\dfrac{a-b}{a^2+b^2}$ (d) $\dfrac{a+b}{a^2+b^2}$

14. Simplify :
$\left(\dfrac{2x+y}{x+y}-1\right)\div\left(1-\dfrac{x}{x+y}\right)$
=____

(a) $\frac{x}{x+y}$ (b) $\frac{x}{x-y}$

(c) $\frac{y}{x}$ (d) $\frac{x}{y}$

15. The square root of $a^2 - 2a + 1$ is __

(a) $\pm(a+1)$ (b) $\pm(a-1)$

(c) $a-1$ (d) $a+1$

16. What should be added to complete the square of $x^4 + 64$?

(a) $8x^2$ (b) $-8x^2$

(c) $16x^2$ (d) $4x^2$

17. The square root of $x^4 + \frac{1}{x^4} + 2$ is __

(a) $\pm\left(x+\frac{1}{x}\right)$ (b) $\pm\left(x^2+\frac{1}{x^2}\right)$

(c) $\pm\left(x-\frac{1}{x}\right)$ (d) $\pm\left(x^2-\frac{1}{x^2}\right)$

18. The square root of $4x^2-12x+9$ is:

(a) $\pm(2x-3)$

(b) $\pm(2x+3)$

(c) $(2x+3)^2$

(d) $(2x-3)^2$

19. L.C.M = ___

(a) $\frac{p(x)\times q(x)}{H.C.F}$ (b) $\frac{p(x).q(x)}{L.C.M}$

(c) $\frac{p(x)}{q(x)\times H.C.F}$ (d) $\frac{q(x)}{p(x)\times H.C.F}$

20. H.C.F. = ___

(a) $\frac{p(x)\times q(x)}{L.C.M}$ (b) $\frac{p(x)\times q(x)}{H.C.F}$

(c) $\frac{p(x)}{q(x)\times L.C.M}$ (d) $\frac{L.C.M}{p(x)\times q(x)}$

21. L.C.M x HCF=

(a) $p(x)\times q(x)$ (b) $p(x)\times H.C.F$

(c) $q(x)\times L.C.M$ (d) None

22. Any unknown expression may be found if ____ of them are known by using the relation L.C.M x H.C.F = p(x) x q(x)

(a) Two

(b) Three

(c) Four

(d) None

ANSWER KEY

1.	a	**2.**	a	**3.**	c	**4.**	b	**5.**	a
6.	a	**7.**	a	**8.**	b	**9.**	c	**10.**	c
11.	c	**12.**	a	**13.**	a	**14.**	d	**15.**	b
16.	c	**17.**	b	**18.**	a	**19.**	a	**20.**	a
21.	a	**22.**	b						

Unit 07 LINEAR EQUATIONS AND INEQUALITIES

Define Linear Equations

A linear equation in one unknown variable x is an equation of the form

$ax + b = 0$, where $a, b \in R$ and $a \neq 0$.

A solution to a linear equation is any replacement or substitution for the variable x that makes the statement true. Two linear equations are said to be equivalent if they have exactly the same solution.

Example

Solve the equation $\frac{3x}{2}-\frac{x-2}{3}=\frac{25}{6}$

Solution

Multiplying each side of the given equation by 6

$$\begin{aligned} 9x-2(x-2) &= 25 \\ \Rightarrow \quad 9x-2x+4 &= 25 \\ \Rightarrow \quad 7x &= 21 \\ \Rightarrow \quad x &= 3 \end{aligned}$$

Check

Substituting x = 3 in original equation,

$$\begin{aligned} \frac{3}{2}(3)-\frac{3-2}{3} &= \frac{25}{6} \\ \frac{9}{2}-\frac{1}{3} &= 25 \\ \frac{25}{6} &= \frac{25}{6} \end{aligned}$$

Since $x = 3$ makes the original statement true, therefore the solution is correct.

Note

Some fractional equations may have no solution.

Example

Solve $\frac{3}{y-1}-2=\frac{3y}{y-1}$, $y \neq 1$

Solution

Multiplying both sides by y − 1, we get

$$\begin{aligned} 3-2(y-1) &= 3y \\ \Rightarrow \quad 3-2y+2 &= 3y \\ \Rightarrow \quad -5y &= -5 \\ \Rightarrow \quad y &= 1 \end{aligned}$$

Check

Substituting y = 1 in the given equation, we have

$$\begin{aligned} \frac{3}{1-1}-2 &= \frac{3(1)}{1-1} \\ \frac{3}{0}-2 &= \frac{3}{0} \end{aligned}$$

But $\frac{3}{0}$ is undefined. So $y=1$ cannot be a solution.

Thus the given equation has not solution.

Example

Solve $\frac{3x-1}{3}-\frac{2x}{x-1}=x$, $x \neq 1$

Solution

Multiplying each side by $3(x-1)$

$$(x-1)(3x-1)-6x = 3x(x-1)$$

$$\Rightarrow 3x^2-4x+1-6x = 3x^2-3x$$

$$\Rightarrow -10x+1 = -3x$$

$$\Rightarrow -7x = -1$$

$$\Rightarrow x = \frac{1}{7}$$

Check

On substituting $x=\frac{1}{7}$ the original equation is verified a true statement. That means the restriction $x \neq 1$ has no effect on the solution because $\frac{1}{7} \neq 1$.

Hence our solution $x=\frac{1}{7}$ is correct.

Define Radical equation?

When the variable in an equation occurs under a radical, the equation is called a radical equation.

Example

Solve the equations

(a) $\sqrt{2x-3}-7=0$

(b) $\sqrt[3]{3x+5}=\sqrt[3]{3x+5}=\sqrt[3]{x-1}$

Solution

(a) To isolate the radical, we can rewrite the given equation as

$$\sqrt{2x-3} = 7$$

$$\Rightarrow 2x-3 = 49 \quad .....$$

$$\Rightarrow 2x = 52 \Rightarrow x = 26$$

Check

Let us substitute $x=26$ in the original equation. Then

$$\sqrt{2(26)-3}-7 = 0$$

$$\sqrt{52-3}-7 = 0$$

$$\sqrt{49}-7 = 0$$

$$0 = 0$$

Hence the solution set is $\{26\}$.

(b) We have

$$\sqrt[3]{3x+5} = \sqrt[3]{x-1}$$

Taking cube of each side

$$\Rightarrow 3x+5 = x-1,$$

$$\Rightarrow 2x=-06 \Rightarrow x=-3$$

Check

We substitute $x = -3$ in the original equation. Then

$$\sqrt[3]{3(-3)+5} = \sqrt[3]{-3-1}$$

$$\sqrt[3]{-9+5} = \sqrt[3]{-4}$$

$$\Rightarrow \sqrt[3]{-4} = \sqrt[3]{-4}$$

Thus $x = -3$ satisfies the original equation.

Here $\sqrt[3]{-4}$ is a real number because we raised each side of the equation to an odd power.

Thus the solution set = $\{-3\}$

Example

Solve and check: $\sqrt{5x-7}-\sqrt{x+10}=0$

Solution

When two terms of a radical equation contain variables in the radicand, we express the equation such that one of these terms is on each side. So we rewrite the equation in this form to get

$$\sqrt{5x-7}-\sqrt{x+10}=0$$

Squaring each side

$$5x-7 = x+10,$$

$$5x - x = 10 + 7$$

$$4x = 17 \Rightarrow x=\frac{17}{4}$$

Check

Substituting $x=\frac{17}{4}$ in original equation

$$\sqrt{5x-7}-\sqrt{x+10} = 0$$

$$\sqrt{5\left(\frac{17}{4}\right)-7}-\sqrt{\frac{17}{4}+10} = 0$$

$$\sqrt{\frac{57}{4}}-\sqrt{\frac{57}{4}} = 0$$

$$0 = 0$$

i.e., $x=\frac{17}{4}$ makes the given equation a true statement.

Thus solution set $=\left\{\frac{17}{4}\right\}$.

Example

Solve $\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$

Solution

$$\sqrt{x+7}+\sqrt{x+2}=\sqrt{6x+13}$$

Squaring both sides we get

$$x+7+x+2+2\sqrt{(x+7)(x+2)}=6x+13$$

$$\Rightarrow \quad 2\sqrt{x^2+9x+14} = 4x+4$$

$$\Rightarrow \quad \sqrt{x^2+9x+14} = 2x+2$$

Squaring again

$$x^2+9x+14 = 4x^2+8x+4$$

$$\Rightarrow \quad 3x^2-x-10 = 0$$

$$\Rightarrow \quad 3x^2-6x+5x-10 = 0$$

$$\Rightarrow \quad 3x(x-2)+5(x-2) = 0$$

$$\Rightarrow \quad (x-2)(3x+5) = 0$$

$$\Rightarrow \quad x=2, \frac{-5}{3}$$

On checking, we see that $x=2$ satisfies the equation, but $x=\frac{-5}{3}$ does not satisfy the equation. So solution set is $\{2\}$ and $x=\frac{-5}{3}$ is an extraneous root.

Exercise 7.1

Q1. Solve the following equations.

i) $\frac{2}{3}x-\frac{1}{2}x=x+\frac{1}{6}$

Sol: Multiplying both sides by 6

$$^2\cancel{6}\left(\frac{2}{\cancel{3}}x\right)-{}^3\cancel{6}\left(\frac{1}{\cancel{2}}x\right)=6(x)+\cancel{6}\left(\frac{1}{\cancel{6}}\right)$$

$$4x-3x=6x+1$$

$$x=6x+1$$

$$-1=6x-x$$

$$-1=5x$$

$$\Rightarrow \quad \boxed{x=-\frac{1}{5}}$$

Check:

Substituting $x=-\frac{1}{5}$ in the given equation

$$\frac{2}{3}\left(-\frac{1}{5}\right)-\frac{1}{2}\left(-\frac{1}{5}\right)=-\frac{1}{5}+\frac{1}{6}$$

$$-\frac{2}{15}+\frac{1}{10}=-\frac{1}{5}+\frac{1}{6}$$

$$\frac{-4+3}{30}=\frac{-6+5}{30}$$

$$-\frac{1}{30}=-\frac{1}{30} \text{ which is true}$$

Hence solution set = $\left\{-\frac{1}{5}\right\}$

ii) $\frac{x-3}{3}-\frac{x-2}{2}=-1$

Multiplying both sides by 6

$$^{2}\cancel{6}\left(\frac{x-3}{\cancel{3}}\right)-{}^{3}\cancel{6}\left(\frac{x-2}{\cancel{2}}\right)=6(-1)$$

$$2x-\cancel{6}-3x+\cancel{6}=-6$$

$$-x=-6$$

$$\boxed{x=6}$$

Check:

Substituting $x=6$ in the given equation

$$\frac{6-3}{3}-\frac{6-2}{2}=-1$$

$$\frac{3}{3}-\frac{4}{2}=-1$$

$$1-2=-1$$

$-1=-1$ which is true, so solution set = $\{6\}$

iii) $\frac{1}{2}\left(x-\frac{1}{6}\right)+\frac{2}{3}=\frac{5}{6}+\frac{1}{3}\left(\frac{1}{2}-3x\right)$

$$\frac{1}{2}x-\frac{1}{12}+\frac{2}{3}=\frac{5}{6}+\frac{1}{6}-\frac{1}{\cancel{3}}(\cancel{3}x)$$

Multiplying both sides by 12

$$\cancel{12}\left(\frac{1}{\cancel{2}}x\right)-\cancel{12}\left(\frac{1}{\cancel{12}}\right)+{}^{4}\cancel{12}\left(\frac{2}{\cancel{3}}\right)={}^{2}\cancel{12}\left(\frac{5}{\cancel{6}}\right)+{}^{2}\cancel{12}\left(\frac{1}{\cancel{6}}\right)-12(x)$$

$$6x-1+8=10+2-12x$$

$$6x+7=12-12x$$

$$6x+12x=12-7$$

$$18x=5$$

$$\boxed{x=\frac{5}{18}}$$

Check:

Substituting $x=\frac{5}{18}$ in the given equation

$$\frac{1}{2}\left(\frac{5}{18}-\frac{1}{6}\right)+\frac{2}{3}=\frac{5}{6}+\frac{1}{3}\left(\frac{1}{2}-\cancel{3}\times\frac{5}{{}^{6}\cancel{18}}\right)$$

$$\frac{1}{2}\left(\frac{5-3}{18}\right)+\frac{2}{3}=\frac{5}{6}+\frac{1}{3}\left(\frac{3-5}{6}\right)$$

$$\frac{1}{2}\left(\frac{2}{18}\right)+\frac{2}{3}=\frac{5}{6}-\frac{2}{18}$$

$$\frac{1+12}{18}=\frac{15-2}{18}$$

$$\frac{13}{18}=\frac{13}{18} \text{ which is true, so}$$

Solution set = $\left\{\frac{5}{18}\right\}$

(iv) $x+\frac{1}{3}=2\left(x-\frac{2}{3}\right)-6x$

$$x+\frac{1}{3}=2x-\frac{4}{3}-6x$$

Multiplying both sides by 3

$$3x+\cancel{3}\times\frac{1}{\cancel{3}}=3(2x)-\cancel{3}\left(\frac{4}{\cancel{3}}\right)-3(6x)$$

$$3x+1=6x-4-18x$$

$$3x+1=-12x-4$$

$$15x=-5$$

$$x=-\frac{5}{15}$$

$$\boxed{x=-\frac{1}{3}}$$

Check:

Substituting $x=-\frac{1}{3}$ in the given equation

$$-\frac{1}{3}+\frac{1}{3}=2\left(-\frac{1}{3}-\frac{2}{3}\right)-\cancel{6}\left(-\frac{1}{\cancel{3}}\right)$$

$$0=2\left(-\frac{\cancel{3}}{\cancel{3}}\right)+2$$

$$0=-2+2$$

$0=0$ which is true, so

Solution set $=\left\{-\frac{1}{3}\right\}$

v) $\frac{5(x-3)}{6}-x=1-\frac{x}{9}$

Multiplying both sides by 18

$$^{3}\cancel{18}\times\frac{5(x-3)}{\cancel{6}}-18x=18-{}^{2}\cancel{18}\left(\frac{x}{\cancel{9}}\right)$$

$$15(x-3)-18x=18-2x$$

$$15x-45-18x=18-2x$$

$$15x-18x+2x=18+45$$

$$-x=63$$

$$\Rightarrow\ x=-63$$

Check:

Substituting $x=-63$ in the given equation

$$\frac{5(-63-3)}{6}-(-63)=1-\frac{(-63)}{9}$$

$$5\frac{\left(\cancel{-66}^{11}\right)}{\cancel{6}}+63=1+\frac{\cancel{63}^{7}}{\cancel{9}}$$

$$-55+63=1+7$$

$8=8$ which is true, so

Solution set $=\{-63\}$

vi) $\frac{x}{3x-6}=2-\frac{2x}{x-2}$

$$\frac{x}{3(x-2)}=2-\frac{2x}{x-2}$$

Multiplying both sides by $3(x-2)$

$$\cancel{3}(x-2)\times\frac{x}{\cancel{3}(x-2)}=2\times3(x-2)-\frac{2x}{x-2}\times3(x-2)$$

$$x=6x-12-6x$$

$$\boxed{x=-12}$$

Check:

Substituting $x=-12$ in the given equation

$$\frac{-12}{3(-12)-6}=2-\frac{2(-12)}{-12-2}$$

$$\frac{-12}{-36-6}=2-\frac{(\cancel{-24})}{\cancel{-14}}$$

$$\frac{-12}{-42}=2-\frac{12}{7}$$

$$\frac{2}{7}=\frac{14-12}{7}$$

$\frac{2}{7}=\frac{2}{7}$ which is true, so

Solution Set $=\{-12\}$

vii) $\frac{2x}{2x+5}=\frac{2}{3}-\frac{5}{4x+10}$, $x\neq-\frac{5}{2}$

$$\frac{2x}{2x+5}=\frac{2}{3}-\frac{5}{2(2x+5)}$$

Multiplying both sides by $6(2x+5)$

$$6\cancel{(2x+5)}\times\frac{2x}{\cancel{2x+5}}=\frac{2}{\cancel{3}}\times{}^{2}\cancel{6}\,(2x+5)-\frac{5}{\cancel{2(2x+5)}}\times{}^{3}\cancel{6}\,\cancel{(2x+5)}$$

$$12x=8x+20-15$$

$$12x-8x=5$$

$$4x=5$$

$$4x=5$$

$$x=\frac{5}{4}$$

Check:

Substituting $x=\frac{5}{4}$ in the given equation

$$\frac{\cancel{2}\left(\frac{5}{\cancel{4}}\right)}{\cancel{2}\left(\frac{5}{\cancel{4}}\right)+5}=\frac{2}{3}-\frac{5}{\cancel{4}\left(\frac{5}{\cancel{4}}\right)+10}$$

$$\frac{\frac{5}{\cancel{2}}}{\frac{5+10}{\cancel{2}}}=\frac{2}{3}-\frac{\cancel{5}}{\cancel{15}}$$

$$\frac{\cancel{5}}{\cancel{15}}=\frac{2}{3}-\frac{1}{3}$$

$\frac{1}{3}=\frac{1}{3}$ which is true, so

Solution set $=\left\{\frac{5}{4}\right\}$

viii) $\frac{2x}{x-1}+\frac{1}{3}=\frac{5}{6}+\frac{2}{x-1},\ x\neq 1$

Multiplying both sides by $6(x-1)$

$$6\cancel{(x-1)}\times\frac{2x}{\cancel{x-1}}+{}^{2}\cancel{6}(x-1)\times\frac{1}{\cancel{3}}$$

$$=\cancel{6}(x-1)\times\frac{5}{\cancel{6}}+6\cancel{(x-1)}\times\frac{2}{\cancel{x-1}}$$

$$12x+2x-2=5x-5+12$$

$$12x+2x-5x=2-5+12$$

$$9x=9$$

$$x=\frac{9}{9}$$

$$\boxed{x=1}$$

Check:

Substituting $x=1$ in the given equator

$$\frac{2(1)}{1-1}+\frac{1}{3}=\frac{5}{6}+\frac{2}{1-1}$$

$$\frac{2}{0}+\frac{1}{3}=\frac{5}{6}+\frac{2}{0}$$

As $\frac{2}{0}$ is undefined, so $x=1$ cannot be a solution thus the given equation has no solution.

ix) $\frac{2}{x^2-1}-\frac{1}{x+1}=\frac{1}{x+1}$, $x\neq\pm1$

$$\frac{2}{(x+1)(x-1)}-\frac{1}{x+1}=\frac{1}{x+1}$$

Multiplying both sides by $(x+1)(x-1)$

$$\cancel{(x+1)(x-1)}\times\frac{2}{\cancel{(x+1)(x-1)}}$$

$$-\cancel{(x+1)}(x-1)\times\frac{1}{\cancel{x+1}}=\frac{1}{\cancel{x+1}}\times\cancel{(x+1)}(x-1)$$

$$2-x+1=x-1$$

$$2+1+1=x+x$$

$$2x=4$$

$$x=\frac{4}{2}$$

$$x=2$$

Check:

Substituting $x=2$ in the given equation

$$\frac{2}{(2)^2-1}-\frac{1}{2+1}=\frac{1}{2+1}$$

$$\frac{2}{4-1}-\frac{1}{3}=\frac{1}{3}$$

$$\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$$

$\frac{1}{3}=\frac{1}{3}$ which is true, so

Solution Set $=\{2\}$

x) $\frac{2}{3x+6}=\frac{1}{6}-\frac{1}{2x+4}$, $x\neq-2$

$$\frac{2}{3(x+2)}=\frac{1}{6}-\frac{1}{2(x+2)}$$

Multiplying both sides by $6(x+2)$

$${}^{2}\cancel{6}\cancel{(x+2)}\times\frac{2}{\cancel{3}\cancel{(x+2)}}=$$

$$\frac{1}{\cancel{6}}\times\cancel{6}(x+2)-\frac{1}{\cancel{2}\cancel{(x+2)}}\times{}^{3}\cancel{6}\cancel{(x+2)}$$

$$4=x+2-3$$

$4 = x - 1$

$4 + 1 = x$

$\boxed{x = 5}$

Check:

Substituting $x = 5$ in the given equation

$$\frac{2}{3(5)+6} = \frac{1}{6} - \frac{1}{2(5)+4}$$

$$\frac{2}{15+6} = \frac{1}{6} - \frac{1}{10+4}$$

$$\frac{2}{21} = \frac{1}{6} - \frac{1}{14}$$

$$\frac{2}{21} = \frac{7-3}{42}$$

$$\frac{2}{21} = \frac{4}{42}$$

$$\frac{2}{21} = \frac{2}{21}$$ which is true, so

Solution Set $= \{5\}$

Q2. Solve each question and check for extraneous solution, if any.

i) $\sqrt{3x+4} = 2$

Squaring both sides

$$\left(\sqrt{3x+4}\right)^2 = (2)^2$$

$3x + 4 = 4$

$3x = 4 - 4$

$3x = 0$

$x = \frac{0}{3}$

$\boxed{x = 0}$

Check:

Substituting $x = 0$ in the given equation

$\sqrt{3x+4} = 2$

$\sqrt{3(0)+4} = 2$

$\sqrt{0+4} = 2$

$\sqrt{4} = 2$

$2 = 2$ which is true, so

Solution Set $= \{0\}$

ii) $\sqrt[3]{2x-4} - 2 = 0$

$\sqrt[3]{2x-4} = 2$

Taking cube of both sides

$$\left(\sqrt[3]{2x-4}\right)^3 = (2)^3$$

$2x - 4 = 8$

$2x = 8 + 4$

$2x = 12$

$x = \frac{\cancel{12}}{\cancel{2}}$

$\boxed{x = 6}$

Check:

Putting $x = 6$ in the given equation.

$\sqrt[3]{2x-4} - 2 = 0$

$\sqrt[3]{2(6)-4} - 2 = 0$

$\sqrt[3]{12-4} - 2 = 0$

$\sqrt[3]{8} - 2 = 0$

$\sqrt[3]{2^3} - 2 = 0$

$2 - 2 = 0$

$0 = 0$ which is true, so

Solution Set $= \{6\}$

iii) $\sqrt{x-3} - 7 = 0$

or $\sqrt{x-3} = 7$

Squaring both sides

$$\left(\sqrt{x-3}\right)^2 = (7)^2$$

$x - 3 = 49$

$x = 49 + 3$

$x = 52$

Check:

Putting $x=52$ in the given equation

$$\sqrt{x-3}-7=0$$
$$\sqrt{52-3}-7=0$$
$$\sqrt{49}-7=0$$
$$7-7=0$$
$$0=0 \quad \text{which true, so}$$

Solution Set $=\{52\}$

iv) $2\sqrt{t+4}=5$

$$\sqrt{t+4}=\frac{5}{2}$$

Squaring both sides

$$\left(\sqrt{t+4}\right)^2=\left(\frac{5}{2}\right)^2$$
$$t+4=\frac{25}{4}$$
$$t=\frac{25}{4}-4$$
$$=\frac{25-16}{4}$$
$$\boxed{t=\frac{9}{4}}$$

Check:

Putting $t=\frac{9}{4}$ in the given equation.

$$2\sqrt{t+4}=5$$
$$2\sqrt{\frac{9}{4}+4}=5$$
$$2\sqrt{\frac{9+16}{4}}=5$$
$$2\sqrt{\frac{25}{4}}=5$$
$$\cancel{2}\left(\frac{5}{\cancel{2}}\right)=5$$
$$5=5 \quad \text{which is true, so}$$

Solution Set $=\left\{\frac{9}{4}\right\}$

v) $\sqrt[3]{2x+3}=\sqrt[3]{x-2}$

Taking cube of both sides

$$\left(\sqrt[3]{2x+3}\right)^3=\left(\sqrt[3]{x-2}\right)^3$$
$$2x+3=x-2$$
$$2x-x=-2-3$$
$$\boxed{x=-5}$$

Check:

Putting $x=-5$ in the given equation.

$$\sqrt[3]{2x+3}=\sqrt[3]{x-2}$$
$$\sqrt[3]{2(-5)+3}=\sqrt[3]{-5-2}$$
$$\sqrt[3]{-10+3}=\sqrt[3]{-7}$$
$$\sqrt[3]{-7}=\sqrt[3]{-7} \quad \text{which is true, so}$$

Solution Set $=\{-5\}$

vi) $\sqrt[3]{2-t}=\sqrt[3]{2t-28}$

Taking cube of both sides

$$\left(\sqrt[3]{2-t}\right)^3=\left(\sqrt[3]{2t-28}\right)^3$$
$$2-t=2t-28$$
$$2+28=2t+t$$
$$3t=30$$
$$t=\frac{\cancel{30}}{\cancel{3}}$$
$$\boxed{t=10}$$

Check:

Putting $t=3$ in the given equation

$$\sqrt[3]{2-t}=\sqrt[3]{2t-28}$$
$$\sqrt[3]{2-10}=\sqrt[3]{2\times10-28}$$

$$\sqrt[3]{-8}=\sqrt[3]{20-28}$$

$\sqrt[3]{-8}=\sqrt[3]{-8}$ which is true, so

Solution Set = $\{10\}$

vii) $\sqrt{2t+6}-\sqrt{2t-5}=0$ or

$$\sqrt{2t+6}=\sqrt{2t-5}$$

Squaring both sides

$$\left(\sqrt{2t+6}\right)^2=\left(\sqrt{2t-5}\right)^2$$

$$2t+6=2t-5$$

$$\cancel{2t}-\cancel{2t}+6=-5$$

$6=-5$ which is not possible, so

Solution Set = $\{\ \}$

viii) $\sqrt{\dfrac{x+1}{2x+5}}=2,\quad x\neq-\dfrac{5}{2}$

Squaring both sides

$$\left(\sqrt{\frac{x+1}{2x+5}}\right)^2=(2)^2$$

$$\frac{x+1}{2x+5}=4$$

$$x+1=4(2x+5)$$

$$x+1=8x+20$$

$$1-20=8x-x$$

$$-19=7x$$

$$\Rightarrow\quad \boxed{x=-\frac{19}{7}}$$

Check:

Putting $x=-\dfrac{19}{7}$ in the given equation

$$\sqrt{\frac{x+1}{2x+5}}=2$$

$$\sqrt{\frac{\frac{-19}{7}+1}{2\left(-\frac{19}{7}\right)+5}}=2$$

$$\sqrt{\frac{\frac{-19+7}{\cancel{7}}}{\frac{-38+35}{\cancel{7}}}}=2$$

$$\sqrt{\frac{\cancel{-12}}{\cancel{-3}}}=2$$

$$\sqrt{4}=2$$

$2=2$ which is true, so

Solution Set = $\left\{\dfrac{-19}{7}\right\}$

Definition

The absolute value of a real number 'a' denoted by $|a|$, is defined as

$$|a|=\begin{cases}a, & \text{if } a\geq 0\\ -a, & \text{if } a<0\end{cases}$$

e.g., $|6|=6$, $|0|=0$ and $|-6|=-(-6)=6$

Some properties of Absolute Value

If $a, b\in$ R, then

(i) $|a|\geq 0$

(ii) $|-a|=|a|$

(iii) $|ab|=|a|.\,|b|$

(iv) $\left|\dfrac{a}{b}\right|=\dfrac{|a|}{|b|},\ b\neq 0$

Example

Solve and check, $|2x+3|=11$

Solution

By definition, depending on whether $(2x+3)$ is positive or negative the given equation is equivalent to

$+(2x+3) = 11$ *or* $\quad -(2x+3)=11$

In practice, these two equations are usually written as

$2x+3 = +11$ or $2x+3 = -11$

$2x = 8$ or $2x = -14$

$x = 4$ or $x = -7$

Check

Substituting $x = 4$, in the original equation, we get

$|2(4)+3| = 11$

i.e., $11 = 11$, true

Now substituting $x = -7$, we have

$|2(-7)+3| = 11$

$|-11| = 11$

$11 = 11$, true

Hence $x = 4, -7$ are the solutions to the given equation.

Or Solution set $= \{-7, 4\}$

Example

Solve $|8x-3| = |4x+5|$

Solution

Since two numbers having the same absolute value are either equal or differ in sign, therefore, the given equation is equivalent to

$8x-3 = 4x+5$ or $8x-3 = -(4x+5)$

$8x-3 = 4x+5$ or $8x-3 = -4x-5$

$8x-4x = 5+3$ or $8x+4x = -5+3$

$4x = 8$ or $12x = -2$

$x = 2$ or $x = -1/6$

On checking we find that $x=2, x=\frac{-1}{6}$ both satisfy the original equation.

Hence the solution set $\left\{-\frac{1}{6}, 2\right\}$.

Sometimes it may happen that the solution(s) obtained do not satisfy the original equation. Such solution(s) (called extraneous) must be rejected. Therefore, it is always advisable to check the solutions in the original equation.

Example 3

Solve and check $|3x+10| = 5x+6$

Solution

The given equation is equivalent to

$\pm(3x+10) = 5x+6$

i.e., $3x+10 = 5x+6$ *or* $3x+10 = -(5x+6)$

$3x+10 = 5x+6$ *or* $3x+10 = -5x-6$

$3x-5x = 6-10$ *or* $3x+5x = -6-10$

$-2x = -4$ *or* $8x = -16$

$x = 2$ or x -2

On checking in the original equation we see that $x = -2$ does not satisfy it.

Hence the only solution is $x = 2$.

Exercise 7.2

Q1. Identify the following statements as True or False.

i) $|x| = 0$ has only one solution. (True)

ii) All absolute value equations have two solutions. (False)

iii) The equation $|x| = 2$ is equivalent to $x = 2$ or $x = -2$. (True)

iv) The equation $|x-4|=-4$ has no solution. (True)

v) The equation $|2x-3|=5$ is equivalent to $2x-3=5$ or $2x+3=5$ (False.)

Q2. Solve for 'x'.

i) $|3x-5|=4$

$\Rightarrow \quad +(3x-5)=4 \quad$ or $\quad -(3x-5)=4$

$3x-5=4 \quad$ or $\quad 3x-5=-4$

$3x=4+5 \quad$ or $\quad 3x=-4+5$

$3x=9 \quad$ or $\quad 3x=1$

$x=3 \quad$ or $\quad x=\frac{1}{3}$

Check:

Substituting $x=3$ in given equation

$$|3(3)-5|=4$$

$$|9-5|=4$$

$$|4|=4$$

$4=4$ which is true

Putting $x=\frac{1}{3}$ in given equation

$$\left|3\left(\frac{1}{3}\right)-5\right|=4$$

$$|1-5|=4$$

$$|-4|=4$$

$4=4$ which is true, so

Solution Set = $\left\{3, \frac{1}{3}\right\}$

ii) $\frac{1}{2}|3x+2|-4=11$

$$\frac{1}{2}|3x+2|=11+4$$

$$\frac{1}{2}|3x+2|=15$$

$$|3x+2|=15\times 2$$

$$|3x+2|=30$$

$+(3x+2)=30 \quad$ or $\quad -(3x+2)=30$

$3x+2=30 \quad$ or $\quad 3x+2=-30$

$3x=30-2 \quad$ or $\quad 3x=-30-2$

$3x=28 \quad$ or $\quad 3x=-32$

$x=\frac{28}{3} \quad$ or $\quad x=\frac{-32}{3}$

Check:

Putting $x=\frac{28}{3}$ in the given equation

$$\frac{1}{2}\left|\cancel{3}\left(\frac{28}{\cancel{3}}\right)+2\right|-4=11$$

$$\frac{1}{2}|28+2|-4=11$$

$$\frac{1}{2}|30|-4=11$$

$$\frac{1}{2}(30)-4=11$$

$$15-4=11$$

$11=11$ which is true

Now putting $x=-\frac{32}{3}$ in the given equation.

$$\frac{1}{2}\left|\cancel{3}\left(-\frac{32}{\cancel{3}}\right)+2\right|-4=11$$

$$\frac{1}{2}|-32+2|-4=11$$

$$\frac{1}{2}|-30|-4=11$$

$$\frac{1}{2}(30)-4=11$$

$15-4=11$

$11=11$ which is true, so

Hence Solution Set $=\left\{\frac{28}{3},-\frac{32}{3}\right\}$

iii) $|2x+5|=11$

$+(2x+5)=11$ or $-(2x+5)=11$

$2x+5=11$ or $2x+5=-11$

$2x=11-5$ or $2x=-11-5$

$2x=6$ or $2x=-16$

$x=\frac{6}{2}$ or $x=\frac{-16}{2}$

$x=3$ or $x=-8$

Check:

Putting $x=3$ in the given equation.

$|2(3)+5|=11$

$|6+5|=11$

$|11|=11$

$11=11$ which is true

Now putting $x=-8$ in the given equation.

$|2(-8)+5|=11$

$|-16+5|=11$

$|-11|=11$

$11=11$ which is true, so

Solution Set $=\{3,-8\}$

iv) $|3+2x|=|6x-7|$

$\frac{|3+2x|}{|6x-7|}=1$

$\left|\frac{3+2x}{6x-7}\right|=1$

$+\left(\frac{3+2x}{6x-7}\right)=1$ or $-\left(\frac{3+2x}{6x-7}\right)=1$

$\frac{3+2x}{6x-7}=1$ or $\frac{3+2x}{6x-7}=-1$

$3+2x=6x-7$ or $3+2x=-6x+7$

$3+7=6x-2x$ or $2x+6x=7-3$

$10=4x$ or $8x=4$

$\Rightarrow x=\frac{10}{4}$ or $x=\frac{4}{8}$

$x=\frac{5}{2}$ or $x=\frac{1}{2}$

Check:

Putting $x=\frac{5}{2}$ in the given equation

$\left|3+\cancel{2}\left(\frac{5}{\cancel{2}}\right)\right|=\left|{}^{3}\cancel{6}\left(\frac{5}{\cancel{2}}\right)-7\right|$

$|3+5|=|15-7|$

$|8|=|8|$

$8=8$ which is true

Now putting $x=\frac{1}{2}$ in the given equation

$\left|3+\cancel{2}\left(\frac{1}{\cancel{2}}\right)\right|=\left|\cancel{6}\left(\frac{1}{\cancel{2}}\right)-7\right|$

$|3+1|=|3-7|$

$|4|=|-4|$

$4=4$ which is true, so

Solution Set $=\left\{\frac{5}{2},\frac{1}{2}\right\}$

v) $|x+2|-3=5-|x-2|$

$|x+2|+|x+2|=5+3$

$2|x+2|=8$

$|x+2|=\frac{\cancel{8}}{\cancel{2}}$

$|x+2|=4$

$$+(x+2)=4 \quad \text{or} \quad -(x+2)=4$$
$$x+2=4 \quad \text{or} \quad x+2=-4$$
$$x=4-2 \quad \text{or} \quad x=-4-2$$
$$x=2 \quad \text{or} \quad x=-6$$

Check:

Putting $x=2$ in the give equation

$$|2+2|-3=5-|2+2|$$
$$|4|-3=5-|4|$$
$$4-3=5-4$$
$$1=1 \quad \text{which is true}$$

Now putting $x=-6$ in the given equation.

$$|-6+2|-3=5-|-6+2|$$
$$|-4|-3=5-|-4|$$
$$4-3=5-4$$
$$1=1 \quad \text{which is true, so}$$

Solution Set = $\{2,-6\}$

vi) $\frac{1}{2}|x+3|+21=9$

$$\frac{1}{2}|x+3|=9-21$$
$$\frac{1}{2}|x+3|=-12$$
$$|x+3|=-24$$

As the value of absolute cannot be negative, so Solution Set = $\{\ \}$

vii) $\left|\frac{3-5x}{4}\right|-\frac{1}{3}=\frac{2}{3}$

$$\left|\frac{3-5x}{4}\right|=\frac{2}{3}+\frac{1}{3}$$
$$\left|\frac{3-5x}{4}\right|=\frac{\cancel{3}}{\cancel{3}}$$
$$\left|\frac{3-5x}{4}\right|=1$$
$$+\left(\frac{3-5x}{4}\right)=1 \quad \text{or} \quad -\left(\frac{3-5x}{4}\right)=1$$
$$\frac{3-5x}{4}=1 \quad \text{or} \quad \frac{3-5x}{4}=-1$$
$$3-5x=4 \quad \text{or} \quad 3-5x=-4$$
$$3-4=5x \quad \text{or} \quad 3+4=5x$$
$$-1=5x \quad \text{or} \quad 7=5x$$
$$x=-\frac{1}{5} \quad \text{or} \quad x=\frac{7}{5}$$

Check:

Putting $x=-\frac{1}{5}$ in the given equation

$$\left|\frac{3-5\left(-\frac{1}{5}\right)}{4}\right|-\frac{1}{3}=\frac{2}{3}$$
$$\left|\frac{3+1}{4}\right|-\frac{1}{3}=\frac{2}{3}$$
$$\left|\frac{4}{4}\right|-\frac{1}{3}=\frac{2}{3}$$
$$|1|-\frac{1}{3}=\frac{2}{3}$$
$$1-\frac{1}{3}=\frac{2}{3}$$
$$\frac{3-1}{3}=\frac{2}{3}$$
$$\frac{2}{3}=\frac{2}{3} \quad \text{which is true,}$$

Now putting $x=\frac{7}{5}$ in the given equation

$$\left|\frac{3-\cancel{5}\left(\frac{7}{\cancel{5}}\right)}{4}\right|-\frac{1}{3}=\frac{2}{3}$$

$$\left|\frac{3-7}{4}\right|-\frac{1}{3}=\frac{2}{3}$$

$$\left|-\frac{4}{4}\right|-\frac{1}{3}=\frac{2}{3}$$

$$|-1|-\frac{1}{3}=\frac{2}{3}$$

$$1-\frac{1}{3}=\frac{2}{3}$$

$\frac{2}{3}=\frac{2}{3}$ which is true

So, solution set = $\left\{-\frac{1}{5},\frac{7}{5}\right\}$

viii) $\left|\frac{x+5}{2-x}\right|=6$

$+\left(\frac{x+5}{2-x}\right)=6$	or	$-\left(\frac{x+5}{2-x}\right)=6$
$\frac{x+5}{2-x}=6$	or	$\frac{x+5}{2-x}=-6$
$x+5=12-6x$	or	$x+5=12+6x$
$x+6x=12-5$	or	$5+12=6x-x$
$7x=7$	or	$17=5x$
$x=1$	or	$x=\frac{17}{5}$

Check:

Putting $x=1$ in the given equation.

$$\left|\frac{1+5}{2-1}\right|=6$$

$$\left|\frac{6}{1}\right|=6$$

$$|6|=6$$

$$6=6$$

Now putting $x=\frac{17}{5}$ in the given equation

$$\left|\frac{\frac{17}{5}+5}{2-\frac{17}{5}}\right|=6$$

$$\left|\frac{\frac{17+25}{\cancel{5}}}{\frac{10-17}{\cancel{5}}}\right|=6$$

$$\left|\frac{42}{-7}\right|=6$$

$$|-6|=6$$

$6=6$ which is true

So, solution set = $\left\{1,\frac{17}{5}\right\}$

Definition of inequality

Let a, b be real numbers, then a is greater than b if the difference a – b is positive and we denote this order relation by the inequality a > b. An equivalent statement is that b is less than a, symbolized by b < a. Similarly, if a – b is negative, then a is less than b and expressed in symbols as a < b.

Properties of Inequalities

1. Law of Trichotomy

For any $a,b \in R$, one and only on of the following statements is true.

a < b or a = b, or a > l

An important special case of thi property is the case for $b = 0$, namely,

$a < 0$ or $a = 0$ or $a > 0$ for any $a \in R$

2. Transitive Property

Let $a, b, c \in$ R.

(i) If $a > b$ and $b > c$, then $a > c$

(ii) If $a < b$ and $b < c$, then $a < c$

3. Additive Closure Property

For $a, b, c \in$ R,

(i) If $a > b$, then $a + c > b + c$

If $a < b$, then $a + c > b + c$

(ii) If $a > 0$ and $b > 0$, then $a + b > 0$

If $a < 0$ and $b < 0$, then $a + b < 0$

4. Multiplicative Property

Let $a, b, c, d \in R$,

(i) If $a > 0$ and $b > 0$, then $ab > 0$,

whereas $a < 0$ and $b < 0 \Rightarrow ab > 0$

(ii) If $a > b$ and $c > 0$, then $ac > bc$

Or if $a < b$ and $c > 0$, then $ac < bc$

(iii) If $a > b$ and $c < 0$, then $ac < bc$

Or if $a < b$ and $c < 0$, then $ac > bc$

The above property (iii) states that the sign of inequality is reversed if each side is multiplied by a negative real number.

(iv) If $a > b$ and $c > d$, then $ac > bd$

Example

Solve $9-7x>19-2x$, where $x \in R$.

Solution

$9-7x>19-2x$

$9-5x>19$

$-5x>10$

$x<-2$

Hence the solution set $= \{x \mid x<-2\}$

Example

Solve $\frac{1}{2}x-\frac{2}{3} \le x+\frac{1}{3}$, where $x \in R$.

Solution

$\frac{1}{2}x-\frac{2}{3} \le x+\frac{1}{3}$

To clear fractions we multiply each side by 6, the L.C.M of 2 and 3 and get

$6\left[\frac{1}{2}x-\frac{2}{3}\right] \le 6\left[x+\frac{1}{3}\right]$

$6\times\frac{1}{2}x-\frac{6\times 2}{3} \le 6x+6\times\frac{1}{3}$

or $3x-4 \le 6x+2$

or $-4-2 \le 6x-3x$

or $-6 \le 3x$

or $-\frac{6}{3} \le x$

$-2 \le x \Rightarrow x \ge -2$

Hence the solution set

$= \{x \mid x \ge -2\}$

Example

Solve the double inequality

$-2 < \frac{1-2x}{3} < 1$, where $x \in R$.

Solution

The given inequality is a double inequality and represents two separate inequalities

$-2<\frac{1-2x}{3}$ and $\frac{1-2x}{3}<1$

$-2<\frac{1-2x}{3}<1$

or $-6<1-2x<3$

or $-7<-2x<2$

or $\frac{7}{2}>x>-1$

i.e., $-1<x<3.5$

Hence S.S $= \{x \mid -1<x<3.5\}$

Example

Solve the inequality

$4x-1 \le 3 \le 7+2x$, where $x \in R$.

Solution

The given inequality holds if and only if both the separate inequalities $4x-1 \le 3$ and $3 \le 7+2x$ hold. We solve each of these inequalities separately.

The first inequality $4x-1 \le 3$ gives

$4x \le 4$ i.e., $x \le 1$ ....(i)

$3 \le 7 + 2x \Rightarrow -4 \le 2x$

i.e.

$-2 \le x \Rightarrow x \ge -2$ ....(ii)

Combining (i) and (ii) we have

$-2 \le x \le 1$

Thus the solution set = $\{x \mid -2 \le x \le 1\}$.

Exercise 7.3

Q1. Solve the following in equalities.

i) $3x + 1 < 5x - 4$

$1 + 4 < 5x - 3x$

$5 < 2x$

$\frac{5}{2} < x$

or $x > \frac{5}{2}$

Solution Set = $\left\{x \mid x > \frac{5}{2}\right\}$

ii) $4x - 10.3 \le 21x - 1.8$

$4x - 21x \le 10.3 - 1.8$

$-17x \le 8.5$

$17x \ge -8.5$

$x \ge -\frac{8.5}{17}$

$x \ge -0.5$

Solution Set = $\{x \mid x \ge -0.5\}$

iii) $4 - \frac{1}{2}x \ge -7 + \frac{1}{4}x$

$4 + 7 \ge \frac{1}{4}x + \frac{1}{2}x$

$11 \ge \frac{x + 2x}{4}$

$11 \ge \frac{3}{4}x$

$\frac{11 \times 4}{3} \ge x$

$\frac{44}{3} \ge x$

or $x \le \frac{44}{3}$

Solution Set = $\left\{x \mid x \le \frac{44}{3}\right\}$

iv) $x - 2(5 - 2x) \ge 6x - 3\frac{1}{2}$

$x - 2(5 - 2x) \ge 6x - \frac{7}{2}$

Multiplying both sides by 2

$2x - 4(5 - 2x) \ge 12x - 7$

$2x - 20 + 8x \ge 12x - 7$

$2x + 8x - 12x \ge 20 - 7$

$-2x \ge 13$

$2x \le -13$

$x \le -\frac{13}{2}$

Solution Set = $\left\{x \mid x \le -\frac{13}{2}\right\}$

v) $\frac{3x+2}{9} - \frac{2x+1}{3} > -1$

Multiplying both sides by 9

$3x + 2 - 3(2x + 1) > -9$

$3x + 2 - 6x - 3 > -9$

$-3x - 1 > -9$

$-3x > 1 - 9$

$-3x > -8$

$x < \frac{-8}{-3}$

$x < \frac{8}{3}$

Solution Set = $\left\{x \mid x < \dfrac{8}{3}\right\}$

vi) $3(2x+1)-2(2x+5)<5(3x-2)$

$6x+3-4x-10<15x-10$

$2x-7<15x-10$

$10-7<15x-2x$

$3<13x$

$\dfrac{3}{13}<x$

or $x>\dfrac{3}{13}$

Solution Set = $\left\{x \mid x>\dfrac{3}{13}\right\}$

vii) $3(x-1)-(x-2)>-2(x+4)$

$3x-3-x+2>-2x-8$

$2x-1>-2x-8$

$2x+2x>1-8$

$4x>-7$

$x>-\dfrac{7}{4}$

Solution Set = $\left\{x \mid x>-\dfrac{7}{4}\right\}$

viii) $2\dfrac{2}{3}x+\dfrac{2}{3}(5x-4)>-\dfrac{1}{3}(8x+7)$

$\dfrac{8}{3}x+\dfrac{2}{3}(5x-4)>-\dfrac{1}{3}(8x+7)$

Multiplying both sides by 3

$8x+2(5x-4)>-(8x+7)$

$8x+10x-8>-8x-7$

$18x-8>-8x-7$

$18x+8x>8-7$

$26x>1$

$x>\dfrac{1}{26}$

Solution Set = $\left\{x \mid x>\dfrac{1}{26}\right\}$

Q2. Solve the following inequalities.

i) $-4<3x+5<8$

$-4<3x+5$ and $3x+5<8$

$-4-5<3x$ and $3x<8-5$

$-9<3x$ and $3x<3$

$-\dfrac{9}{3}<x$ and $x<\dfrac{3}{3}$

$-3<x$ and $x<1$

$-3<x<1$

Solution Set = $\{x \mid -3<x<1\}$

ii) $-5\le\dfrac{4-3x}{2}<1$

$-5\le\dfrac{4-3x}{2}$ and $\dfrac{4-3x}{2}<1$

$-10x\le4-3x$ and $4-3x<2$

$-10-4\le-3x$ and $-3x<2-4$

$-14\le-3x$ and $-3x<-2$

$14\ge3x$ and $3x>2$

$\dfrac{14}{3}\ge x$ and $x>\dfrac{2}{3}$

$\dfrac{14}{3}\ge x>\dfrac{2}{3}$

Solution Set = $\left\{x \mid \dfrac{14}{3}\ge x>\dfrac{2}{3}\right\}$

iii) $-6<\dfrac{x-2}{4}<6$

$-6<\dfrac{x-2}{4}$ and $\dfrac{x-2}{4}<6$

$-24<x-2$ and $x-2<24$

$-24+2<x$ and $x<24+2$

$-22<x$ and $x<26$

Solution Set = $\{x \mid -22<x<26\}$

iv) $3\ge\dfrac{7-x}{2}\ge1$

$$3 \geq \frac{7-x}{2} \quad \text{and} \quad \frac{7-x}{2} \geq 1$$
$$6 \geq 7-x \quad \text{and} \quad 7-x \geq 2$$
$$6-7 \geq -x \quad \text{and} \quad -x \geq 2-7$$
$$-1 \geq -x \quad \text{and} \quad -x \geq -5$$
$$1 \leq x \quad \text{and} \quad x \leq 5$$
$$1 \leq x \leq 5$$

Solution Set $= \{x \mid 1 \leq x \leq 5\}$

v) $3x-10 \leq 5 < x+3$

$$3x-10 \leq 5 \quad \text{and} \quad 5 < x+3$$
$$-5-10 \leq -3x \quad \text{and} \quad -x < 3-5$$
$$-15 \leq -3x \quad \text{and} \quad -x < -2$$
$$15 \geq 3x \quad \text{and} \quad x > 2$$
$$5 \geq x \quad \text{and} \quad x > 2$$
$$5 \geq x > 2$$

Solution Set $= \{x \mid 5 \geq x > 2\}$

vi) $-3 \leq \frac{x-4}{-5} < 4$

$$-3 \leq \frac{x-4}{-5} \quad \text{and} \quad \frac{x-4}{-5} < 4$$
$$\Rightarrow \quad 3 \geq \frac{x-4}{5} \quad \text{and} \quad \frac{x-4}{5} > -4$$
$$15 \geq x-4 \quad \text{and} \quad x-4 > 20$$
$$15+4 \geq x \quad \text{and} \quad x > 4-20$$
$$19 \geq x \quad \text{and} \quad x > -16$$
$$19 \geq x \geq -16$$

Solution Set $= \{x \mid 19 \geq x > -16\}$

vii) $1-2x < 5-x \leq 25-6x$

$$1-2x < 5-x \quad \text{and} \quad 5-x \leq 25-6x$$
$$1-5 \leq 2x-x \quad \text{and} \quad 6x-x \leq 25-5$$
$$-4 < x \quad \text{and} \quad 5x \leq 20$$
$$-4 < x \quad \text{and} \quad x \leq 4$$
$$-4 < x \leq 4$$

Solution Set $= \{x \mid -4 < x \leq 4\}$

viii) $3x-2 < 2x+1 < 4x+17$

$$3x-2 < 2x+1 \quad \text{and} \quad 2x+1 < 4x+17$$
$$-2-1 < 2x-3x \quad \text{and} \quad 2x-4x < 17-1$$
$$-3 < -x \quad \text{and} \quad -2x < 16$$
$$3 > x \quad \text{and} \quad 2x > -16$$
$$3 > x \quad \text{and} \quad x > -8 \qquad 3 > x > -8$$

Solution Set $= \{x \mid 3 > x > -8\}$

Review Exercise 7

Q3. Answer the following short questions.

i) Define a linear inequality in one variable.

Ans. **<u>Linear Inequality in one variable</u>**

Let a, b be real numbers, then a is greater than b if the difference a – b is positive and we denote this order relation by the inequality a > b. An equivalent statement is that b is less than a, symbolized by b < a. Similarly, if a – b is negative, then a is less than b and expressed in symbols as a < b.

ii) State the trichotomy and transitive properties of inequality.

Ans. **<u>Trichotomy Property of inequality</u>**

For any $a, b \in R$, one and only one of the following statements is true.

a < b or a = b, or a > b

<u>Transitive Property of inequality</u>

Let $a, b, c \in R$

i) If a > b and b > c, then a > c

ii) If a > b and b < c, then a < c

iii) **The formula relating degrees Fahrenheit to degrees Celcius is $F=\frac{9}{5}C+32$. For what value of C is $F<0$.**

Ans. According to formula "F" will be zero, if $\frac{9}{5}C+32=0$

$$\frac{9}{5}C=-32$$

$$C=-\frac{32}{9}\times 5$$

$$C=-\frac{160}{9}$$

To get $F<0$ i.e. negative $C<-\frac{160}{9}$

iv) **Seven times the sum of an integer and 12 is at least 50 and at most 60. Write and solve the inequality that expresses this relationship.**

Ans. Let the required integer be x then

$$50\le x+12\le 60$$

$$50\le x+12 \quad \text{and} \quad x+12\le 60$$

$$50-12\le x \quad \text{and} \quad x\le 60-12$$

$$38\le x \quad \text{and} \quad x\le 48$$

$$38\le x\le 48$$

4. **Solve each of the following and check for extraneous solution, if any.**

$$\sqrt{2t+4}=\sqrt{t-1}$$

Squaring both sides

$$\left(\sqrt{2t+4}\right)^2=\left(\sqrt{t-1}\right)^2$$

$$2t+4=t-1$$

$$2t-t=-1-4$$

$$t=-5$$

Check:

$$\sqrt{2t+4}=\sqrt{t-1}$$

$$\sqrt{2(-5)+4}=\sqrt{-5-1}$$

$$\sqrt{-10+4}=\sqrt{-6}$$

$\sqrt{-6}=\sqrt{-6}$ Which is true, so solution Set $=\{-5\}$

ii) $\sqrt{3x-1}-2\sqrt{8-2x}=0$

$$\sqrt{3x-1}=2\sqrt{8-2x}$$

Squaring both sides

$$\left(\sqrt{3x-1}\right)^2=\left(2\sqrt{8-2x}\right)^2$$

$$3x-1=4(8-2x)$$

$$3x-1=32-8x$$

$$3x+8x=32+1$$

$$11x=33$$

$$x=\frac{\cancel{33}}{\cancel{11}}$$

$$x=3$$

Check:

$$\sqrt{3x-1}-2\sqrt{8-2x}=0$$

$$\sqrt{3(3)-1}-2\sqrt{8-2(3)}=0$$

$$\sqrt{9-1}-2\sqrt{8-6}=0$$

$$\sqrt{8}-2\sqrt{2}=0$$

$$2\sqrt{2}-2\sqrt{2}=0$$

$0=0$ Which is true, so

solution set $=\{3\}$

Q5. Solve for x

i) $|3x+14|-2=5x$

$$|3x+14|=5x+2$$

$$\pm(3x+14)=5x+2$$

$$3x+14=\pm(5x+2)$$

$$3x+14=5x+2 \quad \text{or} \quad 3x+14=-5x-2$$

$$3x-5x=2-14 \quad \text{or} \quad 3x+5x=-2-14$$

$$-2x=-12 \quad \text{or} \quad 8x=-16$$

$$x=\frac{12}{2} \quad \text{or} \quad x=-\frac{16}{8}$$

$$x=6 \quad \text{or} \quad x=-2$$

Check:

Put $x=6$ in

$$|3x+14|-2=5x$$
$$|3(6)+14|-2=5(6)$$
$$|18+14|-2=30$$
$$|32|-2=30$$
$$30-2=30$$
$30=30$, which is true

Now put $x=-2$
$$|3(-2)+14|-2\neq 5(-2)$$
$$|-6+14|-2\neq -10$$
$$|8|-2\neq -10$$
$$8-2\neq -10$$
$6\neq -10$ which is not true

So, Solution Set = $\{6\}$

ii) $$\frac{1}{3}|x-3|=\frac{1}{2}|x+2|$$
$$\frac{|x-3|}{|x+2|}=\frac{3}{2}$$
$$\left|\frac{x-3}{x+2}\right|=\frac{3}{2}$$
$$\pm\left(\frac{x-3}{x+2}\right)=\frac{3}{2}$$

or $$\frac{x-3}{x+2}=\pm\frac{3}{2}$$
$$\frac{x-3}{x+2}=\frac{3}{2} \quad \text{or} \quad \frac{x-3}{x+2}=-\frac{3}{2}$$
$$2(x-3)=3(x+2) \text{ or } 2(x-3)=-3(x+2)$$
$$2x-6=3x+6 \text{ or } 2x+3x=6-6$$
$$-x=12 \text{ or } 5x=0$$
$$x=-12 \text{ or } x=0$$

Check:

Put $x=-12$
$$\frac{1}{3}|x-3|=\frac{1}{2}|x+2|$$
$$\frac{1}{3}|-12-3|=\frac{1}{2}|-12+2|$$
$$\frac{1}{3}|-15|=\frac{1}{2}|-10|$$
$$\frac{15}{3}=\frac{10}{2}$$
$5=5$, which is true

Now put $x=0$
$$\frac{1}{3}|0-3|=\frac{1}{2}|0+2|$$
$$\frac{1}{3}|-3|=\frac{1}{2}|2|$$
$$\frac{3}{3}=\frac{2}{2}$$
$1=1$, which is true

So, Solution Set = $\{-12,0\}$

Q6. Solve the following inequality.

i) $$-\frac{1}{3}x+5\leq 1$$
$$-\frac{1}{3}x\leq 1-5$$
$$-\frac{1}{3}x\leq -4$$

Multiplying both sides by -3
$$x\geq 12$$
Solution Set = $\{x / x\geq 12\}$

ii) $$-3<\frac{1-2x}{5}<1$$
$$-3<\frac{1-2x}{5} \text{ and } \frac{1-2x}{5}<1$$

Multiplying both sides by 5
$$-15<1-2x \text{ and } 1-2x<5$$
$$-15-1<-2x \text{ and } -2x<5-1$$
$$-16<-2x \text{ and } -2x<4$$

Multiplying both sides by -1
$$16>2x \text{ and } 2x>-4$$
$$\frac{16}{2}>x \text{ and } x>\frac{-4}{2}$$
$$8>x \text{ and } x>-2$$
$$8>x>-2$$
Solution Set = $\{x / 8>x>-2\}$

Objective

1. *Which of the following is the solution of the inequality* $3-4x \le 11$?
   (a) $x \ge -8$
   (b) $x \ge -2$
   (c) $x \ge \frac{-14}{4}$
   (d) None of these

2. A statement involving any of the symbols <, > or ≤ or ≥ is called:
   (a) Equation (b) Identity
   (c) Inequality (d) Linear equation

3. x = ________ is a solution of the inequality $-2 < x < \frac{3}{2}$
   (a) -5 (b) 3
   (c) 0 (d) $\frac{5}{2}$

4. If x is not larger than 10, then
   (a) $x \ge 8$ (b) $x \le 10$
   (c) $x < 10$ (d) $x > 10$

5. If the capacity c of an elevator is at most 1600 pounds, then __
   (a) $c < 1600$ (b) $c \ge 1600$
   (c) $c \le 1600$ (d) $c > 1600$

6. $x = 0$ is a solution of the inequality
   (a) $x > 0$ (b) $3x + 5 < 0$
   (c) $x + 2 < 0$ (d) $x - 2 < 0$

7. The linear equation in one variable x is:
   (a) $ax + b = 0$
   (b) $ax^2 + bx + c = 0$
   (c) $ax + by + c = 0$
   (d) $ax^2 + by^2 + c = 0$

8. *An inconsistent equation is that* whose solution set is:
   (a) Empty (b) Not empty
   (c) Zero (d) None of these

9. Absolute value of a real number a is defined as:
   (a) $|a| = \begin{cases} a & \text{if } a \ge 0 \\ -a, & \text{if } a < 0 \end{cases}$
   (b) $|a| = \begin{cases} a & \text{if } a \le 0 \\ -a & \text{if } a > 0 \end{cases}$
   (c) $|a| = \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } a < 0 \end{cases}$
   (d) None of these

10. $|x| = a$ is equivalent to:
    (a) $x = a$ or $x = -a$
    (b) $x = \frac{1}{a}$ or $x = \frac{-1}{a}$
    (c) $x = a$ or $x = \frac{-1}{a}$
    (d) None of these

11. A linear inequality in one variable x is:
    (a) $ax + b > 0, \quad a \ne 0$
    (b) $ax^2 + bx + c < 0, a \ne 0$
    (c) $ax + by + c > 0, \quad a \ne 0$
    (d) $ax^2 + by^2 + c < 0, a \ne 0$

12. Law of Trichotomy is …
    $(a, b \in R)$
    (a) $a < b$ or $a = b$ or $a > b$
    (b) $a < b$ or $a = b$
    (c) $a < b$ or $a > b$
    (d) *None of these*

13. Transitive law is____

(a) $a < b$ and $b < c$, then $a < c$
(b) $a > b$ and $b < c$, then $a > c$
(c) $a > b$ and $b < c$, then $a > c$
(d) None of these

14. If $a > b$, $c > 0$ then:

(a) $ac < bc$ (b) $ac > bc$
(c) $ac = bc$ (d) None

15. If $a > b$, $c > 0$ then:

(a) $\frac{a}{c} > \frac{b}{c}$ (b) $\frac{a}{c} < \frac{b}{c}$
(c) $\frac{a}{c} = \frac{b}{c}$ (d) $\frac{b}{c} \neq \frac{b}{c}$

16. If $a > b$, $c < 0$, then:

(a) $\frac{a}{c} < \frac{b}{c}$ (b) $\frac{a}{c} > \frac{b}{c}$
(c) $\frac{a}{c} = \frac{b}{c}$ (d) $\frac{a}{c} \leq \frac{b}{c}$

17. If $a, b \in R$ then:

(a) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ (b) $|ab| = \frac{|a|}{|b|}$
(c) $\left|\frac{b}{a}\right| = \frac{|b|}{|a|}$ (d) None of these

18. When the variable in an equation occurs under a radical, the equation is called a ___equation.

(a) Radical (b) Absolute value
(c) Linear (d) None of these

19. $|x| = 0$ has only ___ solution.

(a) one (b) two
(c) three (d) none of these

20. The equation $|x| = 2$ is equivalent to

(a) $x = 2$ or $x = -2$
(b) $x = -2$ or $x = -2$
(c) $x = 2$ or $x = \frac{1}{2}$
(d) $x = 2$ or $x = \frac{-1}{2}$

21. An __ is equation that is satisfied by every number for which both sides are defined:

(a) Identity (b) Conditional
(c) Inconsistent (c) None

22. An__ equation is an equation whose solution set is the empty set:

(a) Identity (b) Conditional
(c) Inconsistent (d) None

23. A ___ equation is an equation that is satisfied by atleast one number but is not an identity:

(a) Identity (b) Conditional
(c) Inconsistent (d) None

24. $x + 4 = 4 + x$ is ___ equation:

(a) Identity (b) Conditional
(c) Inconsistent (d) None

25. $2x + 1 = 9$ is ___ equation:

(a) Identity (b) Conditional
(c) Inconsistent (d) None

26. $x = x + 5$ is ___ equation:

(a) Identity (b) Conditional
(c) Inconsistent (d) None

27. Equations having exactly the same solution are called ___ equations.

(a) equivalent (b) Linear
(c) Inconsistent (c) None

28. A solution that does not satisfy the original equation is called ____ solution:

(a) Extraneous (b) Root
(c) General (d) None

ANSWER KEY

1.	b	**2.**	c	**3.**	c	**4.**	b	**5.**	c
6.	d	**7.**	a	**8.**	a	**9.**	a	**10.**	a
11.	a	**12.**	a	**13.**	a	**14.**	b	**15.**	a
16.	a	**17.**	a	**18.**	a	**19.**	a	**20.**	a
21.	a	**22.**	c	**23.**	b	**24.**	a	**25.**	b
26.	c	**27.**	a	**28.**	a				

Unit 08

LINEAR GRAPHS AND THEIR APPLICATION

An Ordered Pair of Real Numbers

An ordered pair of real numbers x and y is pair (x, y) in which elements are written in specific order. i.e.,

(i) (x,y) is an ordered pair in which first element is x and second is y such that $(x, y) \neq (y, x)$ for example:

(2, 3) and (3, 2) are two different ordered pairs.

(ii) (x, y) = (m, n) if and only if x = m and y = n.

Cartesian Plane

The Cartesian plane establishes one-to-one correspondence between the set of ordered pairs $R \times R = \{(x, y) \mid x, y \in R\}$ and the points of the Cartesian plane.

In plane two mutually perpendicular straight lines are drawn. The lines are called the coordinate axes. The point O, where the two lines meet is called origin. This plane is called the coordinate plane or the Cartesian plane.

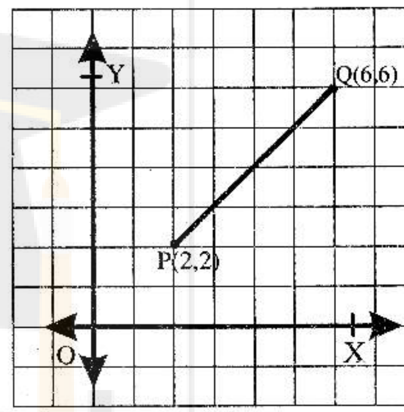
Drawing different geometrical Shapes in Cartesian Plane

(a) Line-Segment

Example:

Let P(2, 2) and Q(6, 6) be two points.

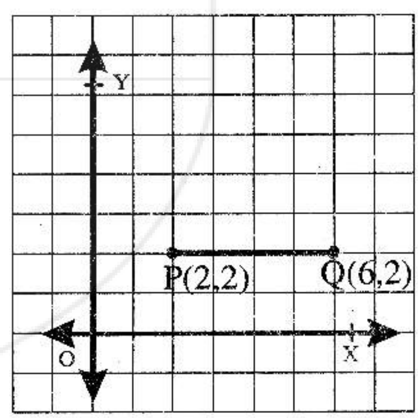
1. Plot points P and Q.
2. Join the points P and Q, we get the line segment PQ. It is represented by $\overline{PQ}$.



Example:

Plot points P(2, 2) and Q(6, 2). By joining them, we get a line segment PQ parallel to x-axis,

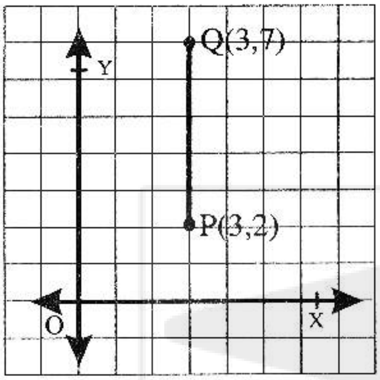
Where ordinate of both points is equal.



Example:

Plot points P(3, 2) and Q(3, 7). By joining them, we get a line segment PQ parallel to y-axis.

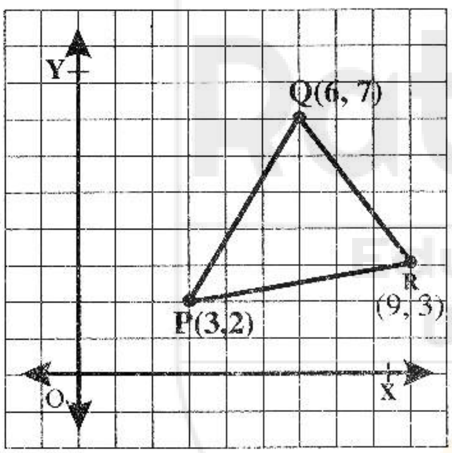
In this graph abscissas of both points are equal.



(b) Triangle

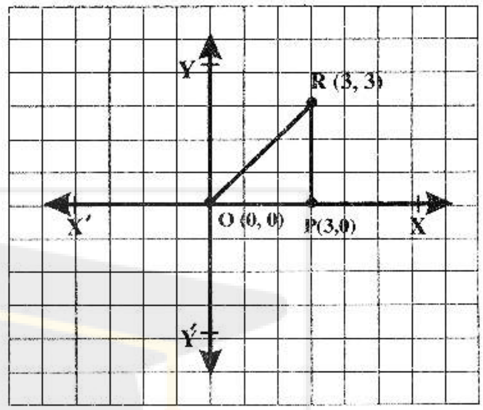
Example:

Plot the points P(3, 2), Q(6, 7) and R(9, 3). By joining them, we get a triangle PQR.



Example:

For points O(0, 0), P(3, 0) and R (3, 3), the triangle OPR is constructed.



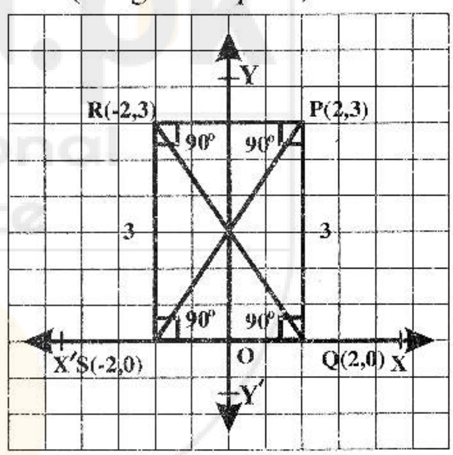
(c) Rectangle

Example

Plot the points P(2, 3), Q(2, 0), S(−2, 0) and R (−2, 3). Joining the points P, Q ,S and R, we get a rectangle PQSR.

Along y-axis,

2 (Length of square) = 1 unit



Construction of a Table for Pairs of Values Satisfying a Linear Equation in Two Variables.

Let 2x + y = 1 (i)

Be a linear equation in two variables x and y.

The ordered pair (x, y) satisfies the equation and by varying x, corresponding y is obtained.

We express (i) in the form

$$\boxed{y = 1 - 2x} \qquad \text{(ii)}$$

The pairs (x, y) which satisfy (ii) are tabulated below.

x	y	(x, y)
–1	3	(–1, 3)
0	1	(0,1)
1	–1	(1,–1)
3	–5	(3, –5)

$at\ x=-1, y=(-2)(-1)+1=2+1=3$

$at\ x=0, y=(-2)(0)+1=0+1=1$

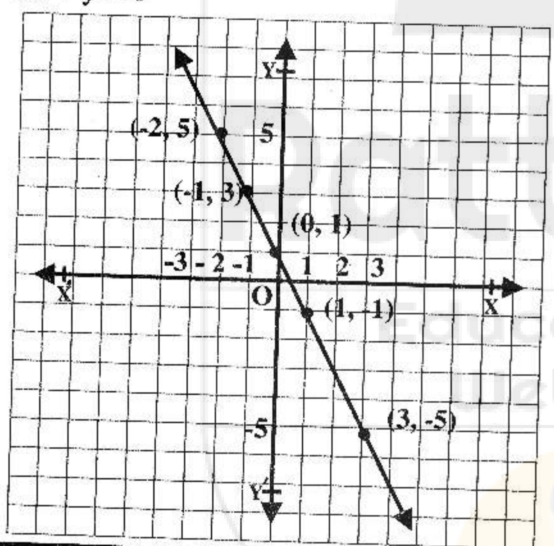
$at\ x=1, y=(-2)(1)+1=-2+1=-1$

$at\ x=3, y=-2(3)+1=-6+1=-5$

Similarly all the points can be computed, the ordered pairs of which do satisfy the equation (i)

Plotting the points to get the graph

Now we plot the points obtained in the table. Joining these points we get the graph of the equation. The graph of $2x + y = 1$



Example:

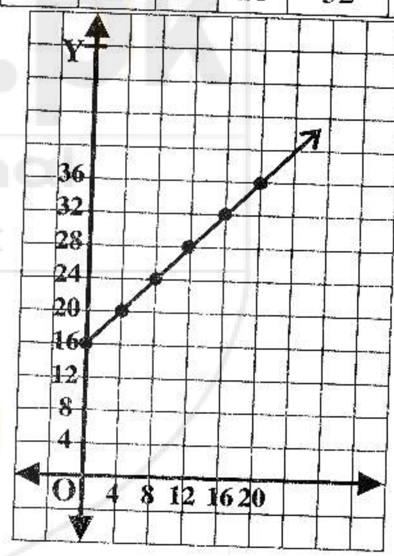
Equation $y = x + 16$ shows the relationship between the age of father and son i.e., if the age of son is x, then the father's age is y. Draw the graph.

Solution:

We know that $y = x + 16$

Table of points for equation is given as:

x	0	4	8	12	16	20
y	16	20	24	28	32	36



Exercise 8.1

1. Determine the quadrant of the coordinate plane in which the following points lie.

Ans. (i) P (–4, 3) II quadrant

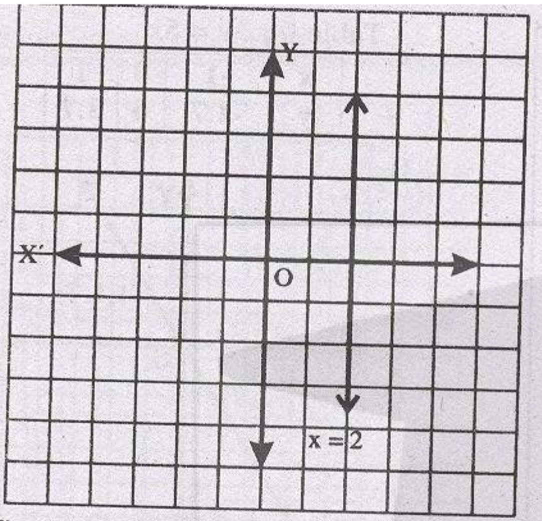
(ii) Q (–5, –2) III quadrant

(iii) P (2, 2) I quadrant

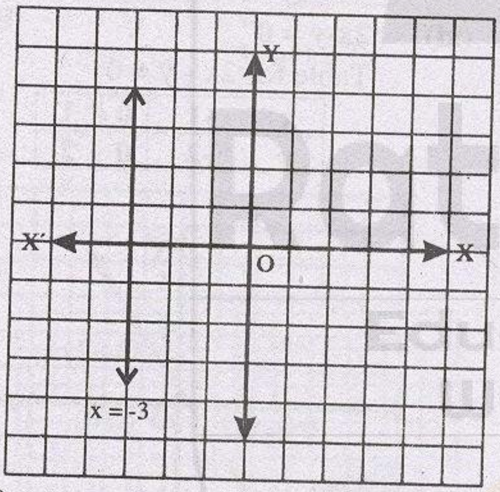
(iv) S(2, –6) IV quadrant

2. Draw the graph of each of the following.

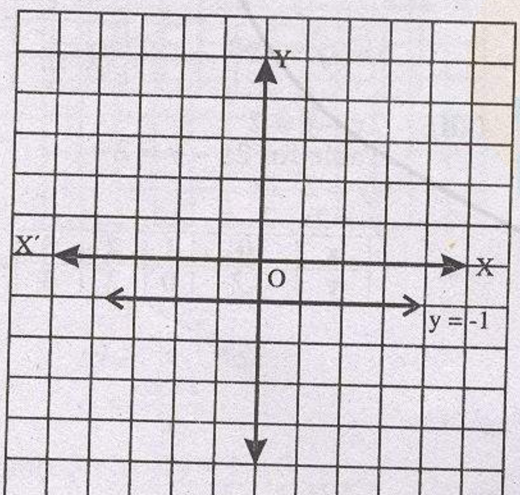
(i) $x=2$



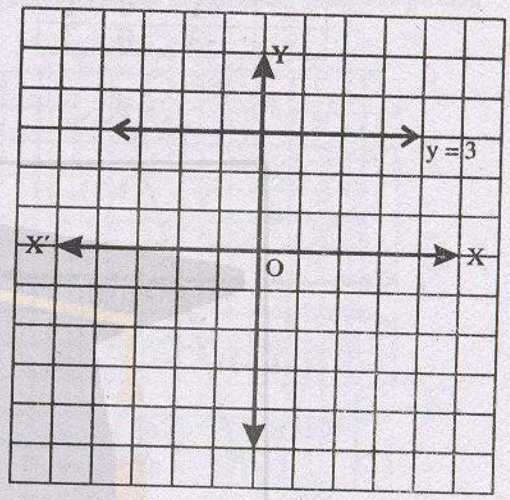
(ii) $x = -3$



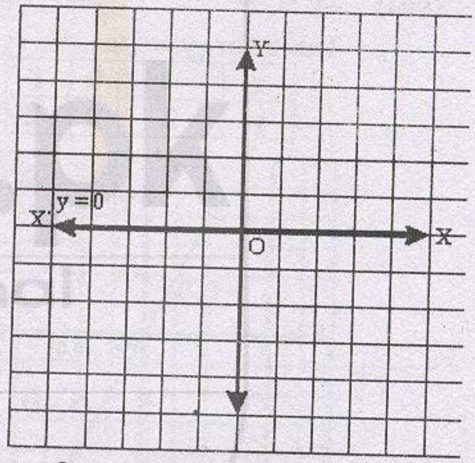
(iii) $y = -1$



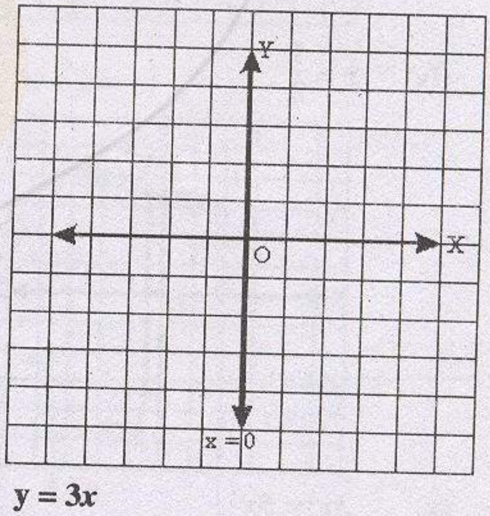
(iv) $y = 3$



(v) $y = 0$



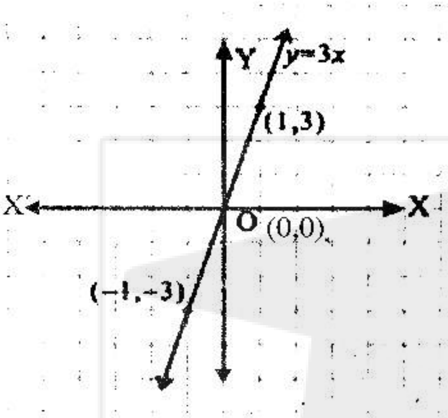
(vi) $x = 0$



(vii) $y = 3x$

Table for y = 3x

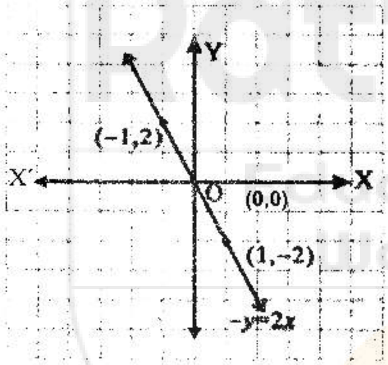
x	-1	0	1
y	-3	0	3



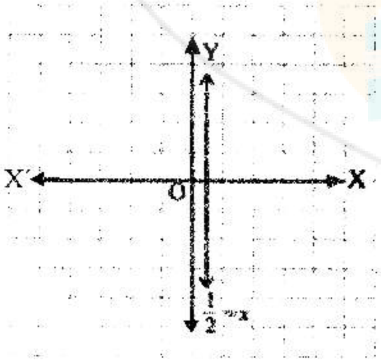
(viii) $-y = 2x$

Table for -y = 2x

x	-1	0	1
y	2	0	-2



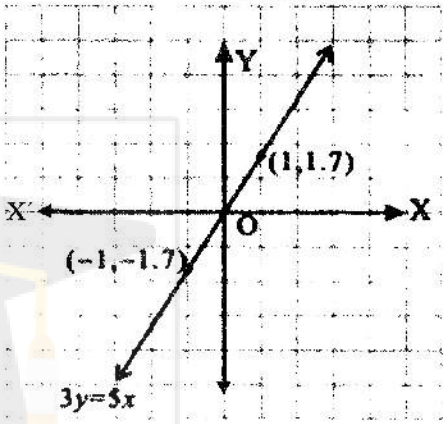
(ix) $x = \frac{1}{2}$



(x) $3y = 5x$

Table for 3y = 5x

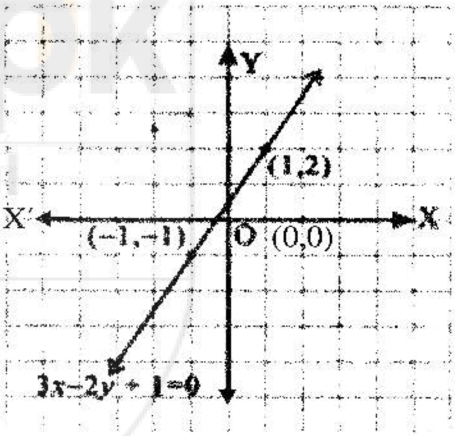
x	-1	0	1
y	-1.7	0	1.7



(xi) **2x-y = 0**

Table for 2x - y = 0

x	-1	0	1
y	-2	0	2



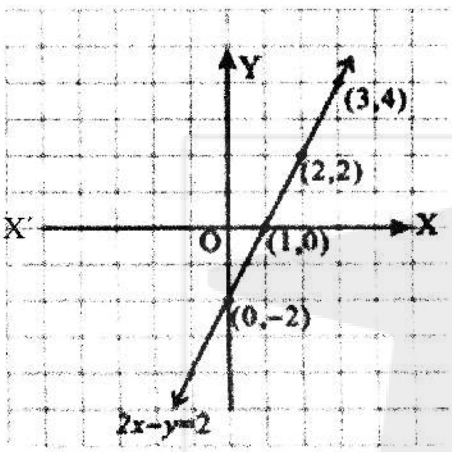
(xii) $2x - y = 2$

Table for 2x − y = 2

-y = 2 -2x

y = 2x -2

x	0	1	2	3
y	-2	0	2	4



(xiii) $x - 3y + 1 = 0$

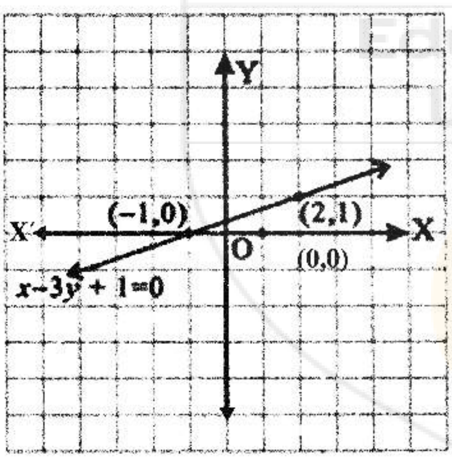
Table for $x - 3y + 1 = 0$

$-3y = -x - 1$

$3y = x + 1$

$y = \frac{x+1}{3}$

x	**-1**	**2**
y	**0**	**1**



(xiv) $3x - 2y + 1 = 0$

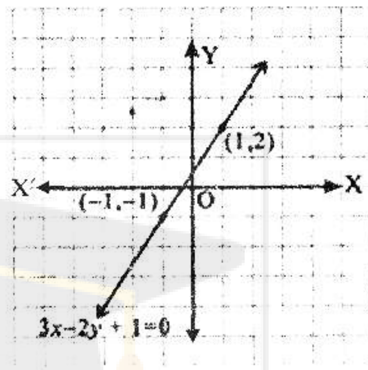
$-2y = -3x - 1$

$2y = 3x + 1$

$y = \frac{3x+1}{2}$

Table for $3x - 2y + 1 = 0$

x	**-1**	**1**
y	**-1**	**2**



Q.3 Are the following lines:

(i) Parallel to x–axis

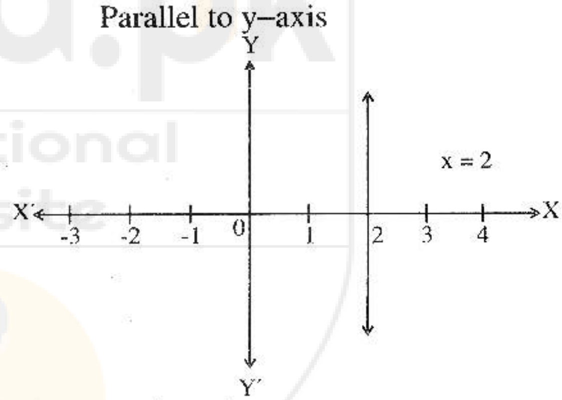
(ii) Parallel to y–axis

(i) $2x - 1 = 3$

$2x = 3 + 1$

$x = \frac{4}{2} = 2$

Parallel to y–axis

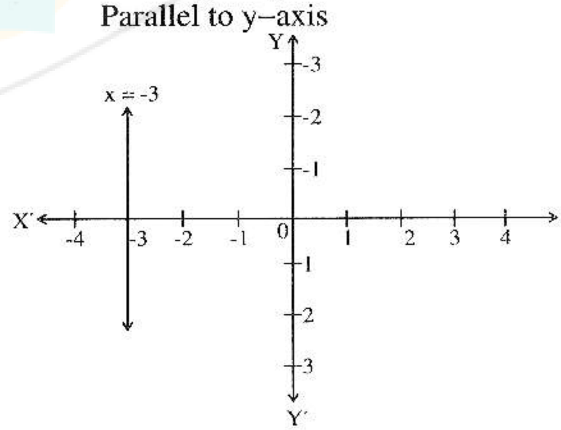


(ii) $x + 2 = -1$

$\Rightarrow \quad x = -1 - 2$

$x = -3$

Parallel to y–axis

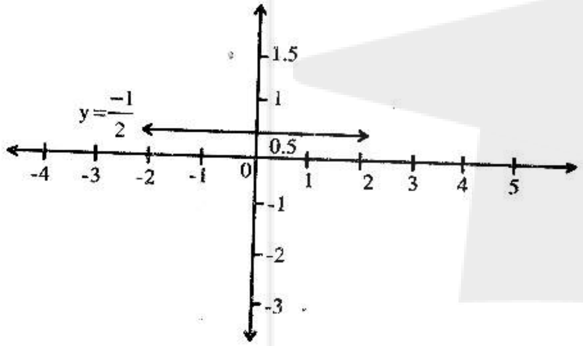


(iii) $2y + 3 = 2$

$\Rightarrow \quad 2y = 2 - 3$

$y = -\dfrac{1}{2}$

Parallel to x–axis



(iv) $x + y = 0$

$\Rightarrow \quad x = -y$

neither

(v) $2x - 2y = 0$

$2x = 2y$

$x = y$

neither

Q.4 Find the value of m and c of the following lines by expressing them in the form y = mx + c

(a) $\mathbf{x - 2y = -2}$

$$-2y = -2 - x$$
$$2y = 2 + x$$
$$y = \frac{2+x}{2}$$
$$y = 1 + \frac{1}{2}x$$
$$y = \frac{1}{2}x + 1 \;\ldots\ldots(1)$$
$$y = mx + c \;\ldots..(2)$$

comparing (1) and (2) we get

$m = \dfrac{1}{2}$ and $c = 1$

(b) $\mathbf{2x + 3y - 1 = 0}$

$$3y = -2x + 1$$
$$y = \frac{-2x+1}{3}$$
$$y = \frac{-2}{3}x + \frac{1}{3} \;\ldots.(1)$$
$$y = mx + c \;\ldots\ldots.(2)$$

comparing (1) and (2) we get

$m = \dfrac{-2}{3}$ and $c = \dfrac{1}{3}$

(c) $\mathbf{3x + y - 1 = 0}$

$$y = -3x + 1 \ldots..(1)$$

Also $y = mx + c \ldots\ldots(2)$

Comparing (1) and (2)

$m = -3$ and $c = 1$

(d) $\mathbf{2x - y = 7}$

$$-y = 7 - 2x$$
$$y = -7 + 2x$$
$$y = 2x - 7 \ldots..(1)$$

also $y = mx + c \;\ldots\ldots(2)$

comparing (1) and (2)

$m = 2$ and $c = -7$

(e) $\mathbf{3 - 2x + y = 0}$

$$y = -3 + 2x$$
$$y = 2x - 3 \;\ldots.(1)$$

Also $y = mx + c \ldots..(2)$

Comparing (1) and (2) we get

$m = 2$ and $c = -3$

(f) $\mathbf{2x = y + 3}$

$$y = 2x - 3 \;\ldots\ldots.(1)$$

Also $y = mx + c \ldots..(2)$

Comparing (1) and (2) we get

$m = 2$ and $c = -3$

Q.5 Verify whether the following points lies on the line 2x – y + 1 = 0 or not.

Ans. $2x - y + 1 = 0$

(i) $(2, 3) \Rightarrow x = 2, y = 3$

$2x - y + 1 = 0$

$\Rightarrow 2(2) - 3 + 1 = 0$

$4 - 3 + 1 \neq 0$

$2 \neq 0$ Point (2,3) does not lie on the line

(ii) $(0, 0) \Rightarrow x = 0, y = 0$

$2x - y + 1 = 0$

$\Rightarrow 2(0) - 0 + 1 = 0$

$1 \neq 0$

Point (0,0) does not lie on the line

(iii) $(-1, 1) \Rightarrow x = -1, y = 1$

$2x - y + 1 = 0$

$\Rightarrow 2(-1) -(1) + 1 - 0 = 0$

$-2 -1 + 1 = 0$

$-2 \neq 0$

Point (-1,1) does not lie on the line

(iv) $(2, 5) \Rightarrow x = 2, y = 5$

$2x - y + 1 = 0$

$\Rightarrow 2(2) - 5 + 1 = 0$

$4 - 5 + 1 = 0$

$-1 + 1 = 0$

$0 = 0$

Yes the Point (2,5) lies on the line

(v) $(5, 3) \Rightarrow x = 5, y = 3$

$2x - y + 1 = 0$

$\Rightarrow 2(5) - 3 + 1 = 0$

$10 - 2 = 0$

$8 \neq 0$

The point (5, 3) does not lie on the line

(a) Example: (Kilometre (Km) and Mile (M) Graphs)

To draw the graph between kilometre (Km) and Miles (M), we use the following relation:

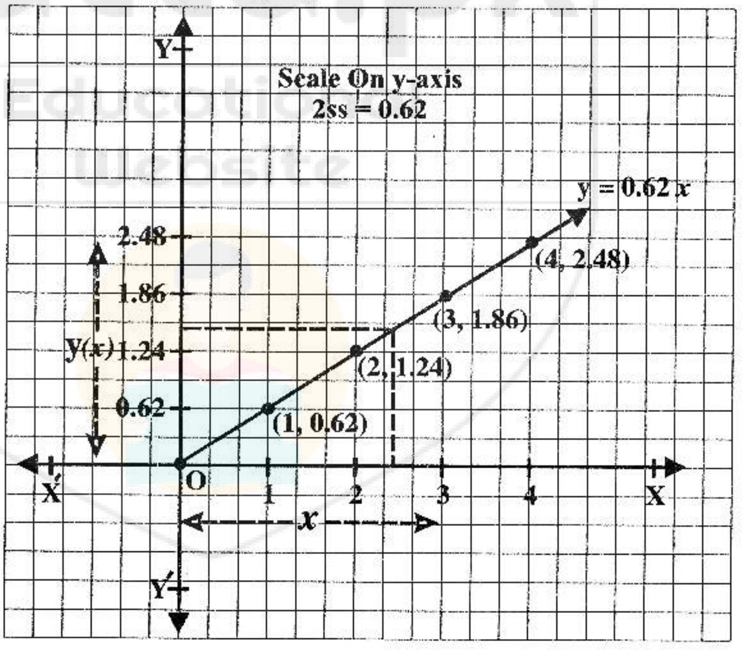
One kilometre = 0.62 miles, (approximately)

And One mile = 1.6 km (approximately)

(i) The relation of mile against kilometre is given by the linear equation,

$y = 0.62\,x,$

If y is a mile and x is a kilometre, then we tabulate the ordered pairs (x, y) as below;



x	0	1	2	3	4
y	0	0.62	1.24	1.86	2.48

The ordered pairs (x, y) corresponding to $y = 0.62x$ are represented in the Cartesian plane. By joining them we get the desired graph of miles against kilometers.

For each quantity of kilometre x along x-axis their corresponding mile along y-axis.

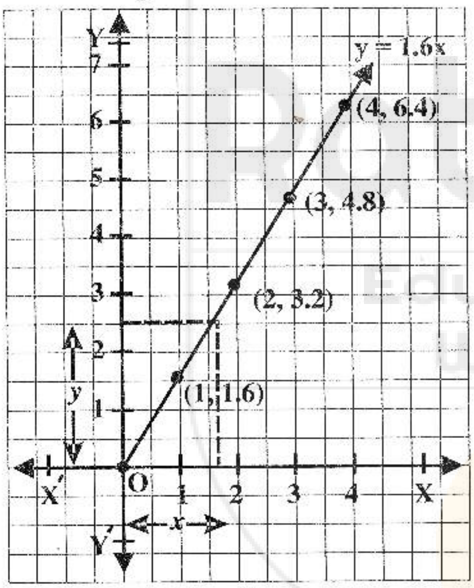
(ii) The conversion graph of kilometer against mile is given by

$$y = 1.6x \qquad \text{(approximately)}$$

If y represents kilometers and x a mile, then the values x and y are tabulated as:

x	0	1	2	3	4 ....
y	0	1.6	3.2	4.8	6.4 ...

We plot the points in the xy-Plane corresponding to the ordered pairs.
(0,0), (1, 1.6), (2, 3.2) (3, 4.8) and (4, 6.4) as shown in figure.



By joining the points we actually find the conversion graph of kilometers against miles.

(b) Conversion Graph of Hectares and Acres

(i) The relation between Hectare and Acre is defined as:

$$\text{Hectare} = \frac{640}{259} \text{ Acres}$$

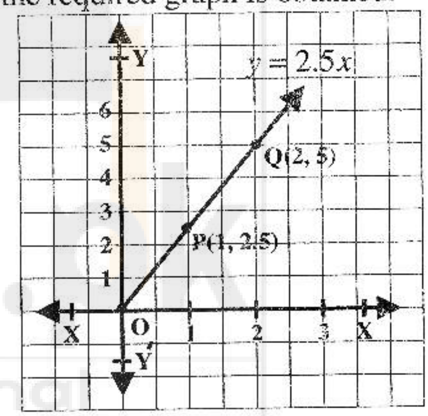
$$= 2.5 \text{ Acres (approximately)}$$

In case when hectare = x and acre = y, then relation between them is given by the equation, $y = 2.5x$

If x is represented as hectare along the horizontal axis and y as Acre along y-axis, the values are tabulated below:

x	0	1	2	3	4 ....
y	0	2.5	5.0	7.5	10 ....

The ordered pairs (0, 0), (1, 2.5), (2,5) etc., are plotted as points in the xy-plane as below and by joining the points the required graph is obtained:



b – (i)

(ii) Now the conversion graph

$$\text{Acre} = \frac{1}{2.5} \text{ Hectare is simplified as,}$$

$$\text{Acre} = \frac{10}{25} \text{ Hectare}$$

$$= 0.4 \text{ Hectare (approximately)}$$

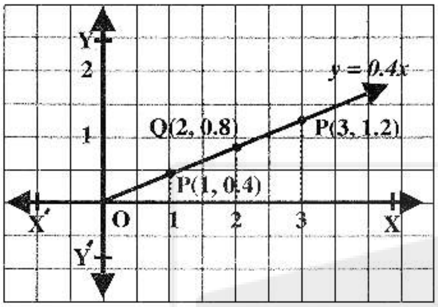
If Acre is measured along x-axis and hectare along y-axis then

$$y = 0.4x$$

The ordered pairs are tabulated in the following table:

x	0	1	2	3 ...
y	0	0.4	0.8	1.2 ...

The corresponding ordered pairs (0, 0), (1, 0.4), (2, 0.8) etc., are plotted in the xy-plane, join of which will form the graph of (b)-ii as a conversion graph of (b)-i:



b – (ii)

(c) Conversion Graph of Degrees Celsius and Degrees Fahrenheit

(i) The relation between Celsius (C) and degree Fahrenheit (F) is given by

$$F = \frac{9}{5}C + 32$$

The value of F at C = 0 is obtained as

$$F = \frac{9}{5}\times 0 + 32 = 0 + 32 = 32$$

Similarly,

$$F = \frac{9}{5}\times 10 + 32 = 18 + 32 = 50,$$

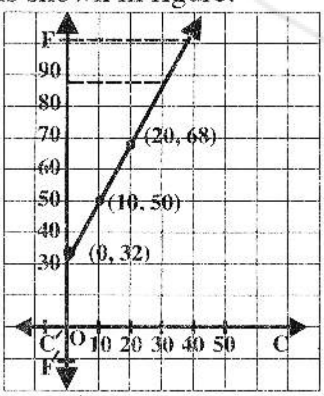
$$F = \frac{9}{5}\times 20 + 32 = 36 + 32 = 68,$$

$$F = \frac{9}{5}\times 100 + 32 = 180 + 32 = 212$$

We tabulate the values of C and F.

C	0°	10°	20°	50°	100°…
F	32°	50°	68°	122°	212°…

The conversion graph of F with respect to C is shown in figure.



10° = length of square

(d) Conversion graph of US$ and Pakistani Currency

The daily News, on a particular day informed the conversion rate of Pakistani currency to the US$ currency as.

1 US$ = 66.46 Rupees

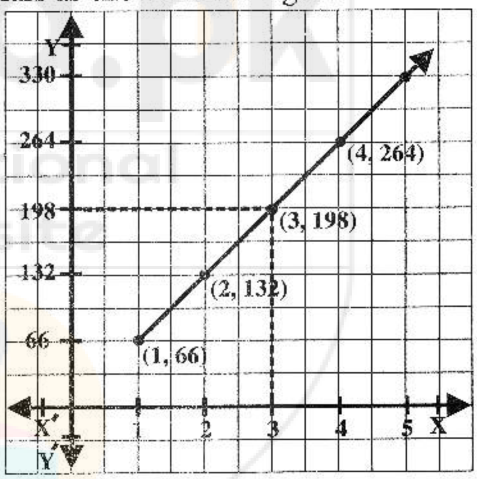
If the Pakistani currency y is an expression of US$ x, expressed under the rule

$y = 66.46\,x \cong 66x$ (approximately)

Then draw the conversion graph.

x	1	2	3	4 …
y	66	132	198	264 …

Plotting the points corresponding to the ordered pairs (x, y) from the above table and joining them provides the currency linear graph of rupees against dollars as shown in the figure.



Conversion graph $x = \frac{1}{66}y$ of y = 66x can be shown by interchanging x-axis to y-axis and vice versa.

Exercise 8.2

Q.1 Draw the conversion graph between 1 litre and gallons using the relation 9 litres = 2 gallons (approximately) and taking litres along horizontal axis and gallons alongs vertical axis. From the graph, read:

(i) The number of gallons in 18 litres

(ii) The number of litres in 8 gallons

Ans.

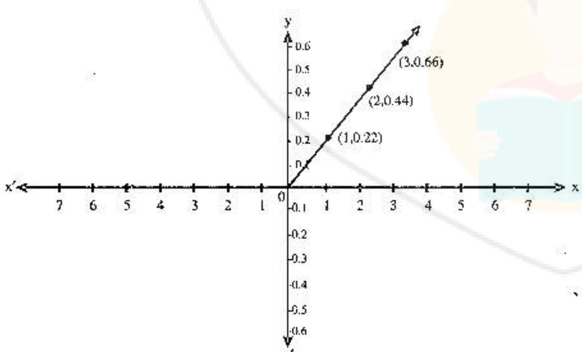
9 litres = 2 gallons	
1 litre = $\frac{2}{9}$ gallons	1 gallon = $\frac{9}{2}$ liter
1 litre= 0.222gallons	1 gallon=4.5 liter

Let gallon be represent by y and litre be x

$$y = 0.222x$$

Table of values

x	0	1	2	3
y	0	0.222	0.444	0.666



(i) Number of gallons in litre

$y = 0.222(18) = \quad = 4$ gallons

(ii) Number of litres in 8 gallons

$\frac{9}{2}(8) = 36$ litres

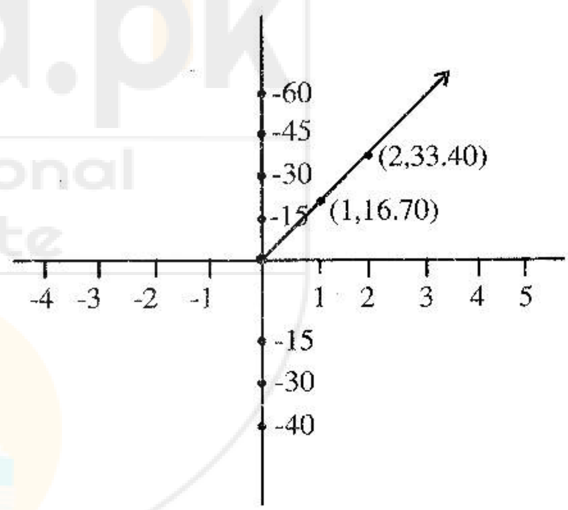
Q.2 On 15.03.2008 the exchange rate of Pakistani currency and Saudi Riyal was as, under 1 S. Riyal = 16.70 rupees.

If Pakistani currency y is an expression of S. Riyal x, expressed under the rule y = 16.70x then draw conversion graph between two currencies by taking S. Riyal along x–axis.

Ans. y = 16.70x.

Table of values

x	0	1	2
y	0	16.70	33.40



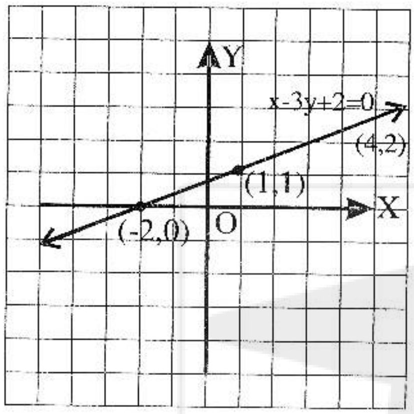
Q.3 Sketch the graph of each of the following lines:

Ans.

(i) $x - 3y + 2 = 0 \Rightarrow -3y = -x - 2$

$$y = \frac{x+2}{3}$$

x	–2	1	4
y	0	1	2



(ii) $\mathbf{3x - 2y - 1 = 0}$

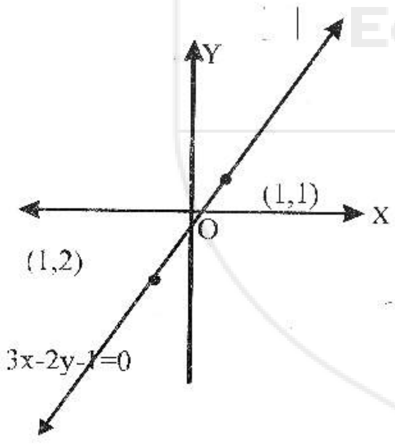
$\mathbf{-2y = 1 - 3x}$

$\mathbf{2y = -1 + 3x}$

$\mathbf{y = \dfrac{3x-1}{2}}$

Table of values

x	−1	1	3
y	−2	1	4



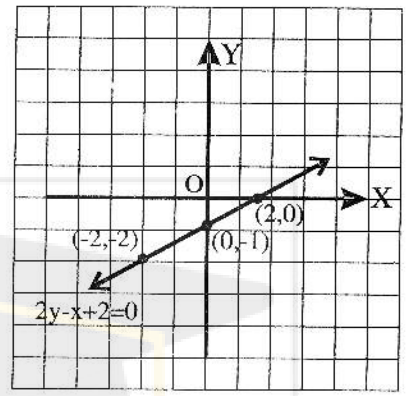
(iii) $\mathbf{2y - x + 2 = 0}$

$\mathbf{2y = x - 2}$

$\mathbf{y = \dfrac{x-2}{2}}$

Table of values

x	−2	0	2
y	−2	−1	0

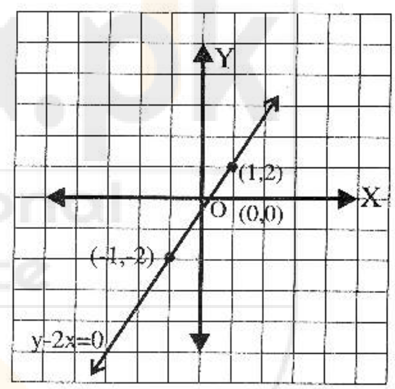


(iv) $\mathbf{y - 2x = 0}$

$\mathbf{y = 2x}$

Table of values

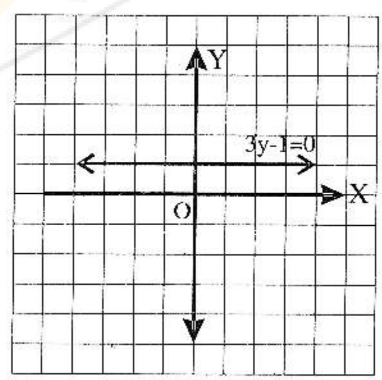
x	−1	0	1
y	−2	0	2



(v) $\mathbf{3y - 1 = 0}$

$\mathbf{3y = 1}$

$\mathbf{y = \dfrac{1}{3}}$



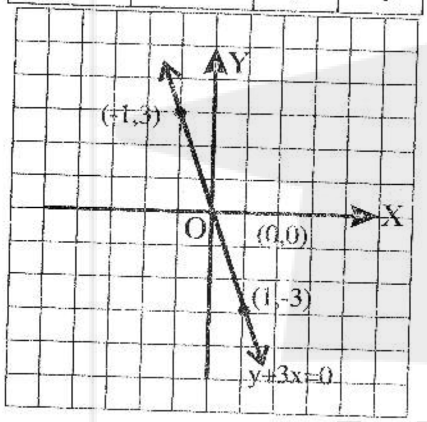
3 (length of square) = 1 unit

(vi) $\mathbf{y + 3x = 0}$

$\mathbf{y = -3x}$

Table of values

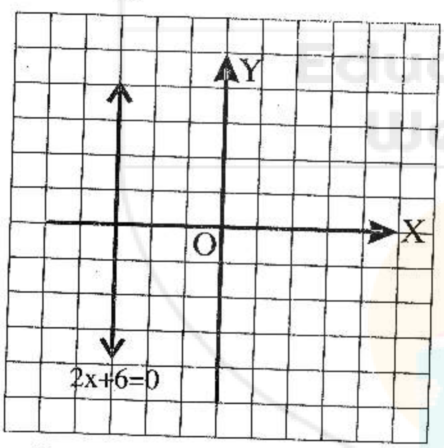
x	−1	0	1
y	3	0	−3



(vii) $\mathbf{2x + 6 = 0}$

$\mathbf{2x = -6}$

$\mathbf{x} = \dfrac{-6}{2} = -3$



Q.4 Draw the graph for following relations:

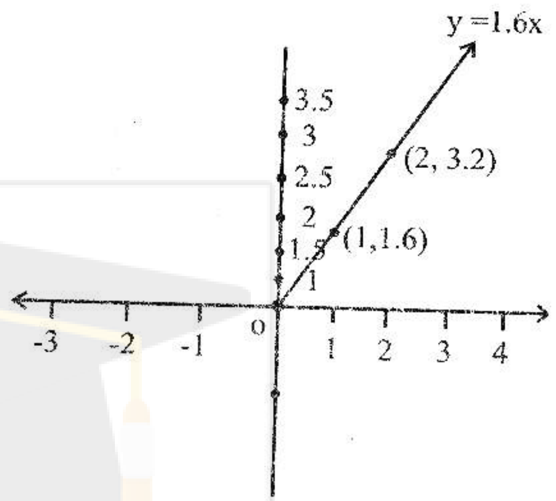
(i) One mile = 1.6 km

Let mile be represented by y and km by x:

$\mathbf{y = 1.6\,x}$

Table of values

X	1	2	3
y	1.6	3.2	4.8

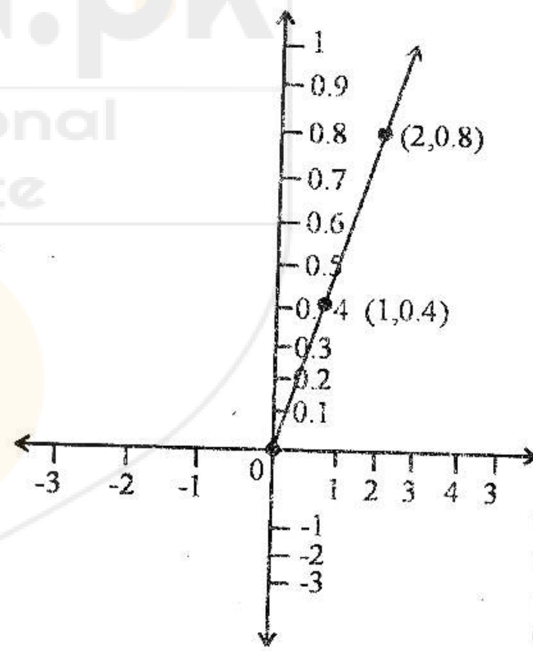


(ii) One acre =0.4 Hectare

$\mathbf{y = 0.4x}$

Table of values

x	0	1	2
y	0	0.4	0.8



(iii) $F = \dfrac{9}{5}c + 32$

The value of F at C = 0 is obtained

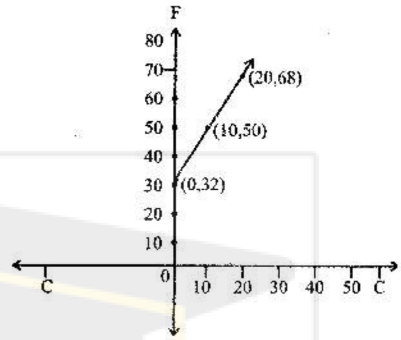
As $F = \dfrac{9}{5} \times 0 + 32 = 0 + 32 = 32$

$$F=\frac{9}{5}\times10+32=36+32=68$$

$$F=\frac{9}{5}\times100+32=180+32=212$$

We tabulate the values of C and F

C	0°	10°	20°	50°	100°
F	32°	50°	68°	122°	212°



Graphical Solution of Linear equations in Two Variables

We solve here simultaneous linear equations in two variables by graphical method

Let the system of equations be,

$2x-y=3,$ ……(i)

$x+3y=3.$ ……(ii)

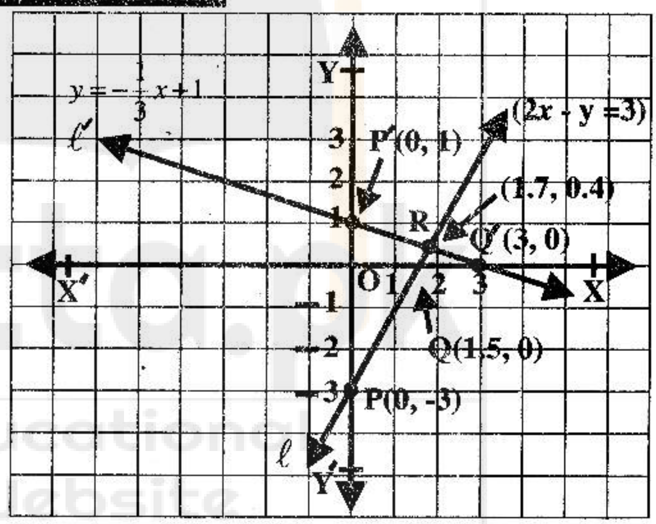
Table of Values

$y=2x-3$

x	0	1.5
y	-3	0

$y=-\frac{1}{3}x+1$

x	0	3	
y	1	0	



By plotting the points we get the following graph.

The solution of the system is the point R where the lines ℓ and ℓ' meet at, i.e., R(1.7,0.4) such that $x=1.7$ and $y=0.4$

Example

Solve graphically, the following linear system of two equations in two variables x and y;

$x+2y=3,$ ....(i)

$x-y=2.$ ....(ii)

Solution

The equations (i) and (ii) are represented graphically with the help of their points of intersection with the coordinate axes of the same co-ordinate plane.

The points of intersections of the lines representing equation (i) and (ii) are given in the following table:

$y=-\frac{1}{2}x+\frac{3}{2}$

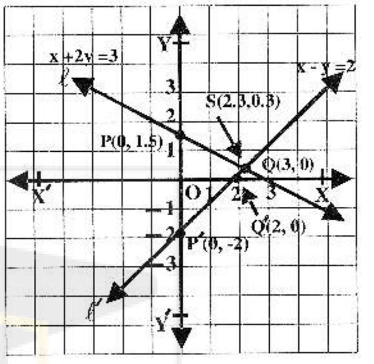
x	0	3
y	1.5	0

$y=x-2$

x	0	2
y	-2	0

The points P(0, 1.5) and Q (3, 0) of equation (i) are plotted in the plane and the corresponding line $\ell : x+2y=3$ is traced by joining P and Q.

Similarly, the line $\ell' : x - y = 2$ of (ii) is obtained by plotting the points $P'(0,-2)$ and $Q'(2,0)$ in the plane and joining them to trace the line ℓ' as below:



The common point S(2.3, 0.3) on both the lines ℓ and ℓ' is the required solution of the system.

Exercise 8.3

Solve the following pair of equations in x and y graphically.

Q.1 x + y = 0 and 2x − y + 3 = 0

Solution: $\Rightarrow$ **y = 0 − x**

Table of values

x	−3	−2	−1	0	1	2
y	3	2	1	0	−1	−2

$2x - y + 3 = 3$

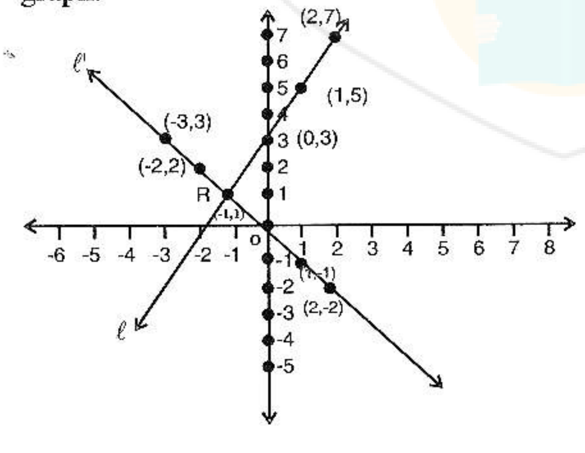
$\Rightarrow \quad -y = -3 - 2x$

$y = 3 + 2x$

Table of values

x	−2	−1	0	1	2
y	−1	1	3	5	7

By plotting the points we get the following graph.



The solution of the system is the point R where the lines ℓ and ℓ' meet at R(−1,1) such that x = −1 and y = 1

Q.2 x − y + 1 = 0 and x − 2y = −1

Solution: y = x + 1

Table of values,

x	−4	−3	−2	−1	0	1	2
y	-3	-2	-1	0	1	2	3

$x - 2y = -1$

$-2y = -1 - x$

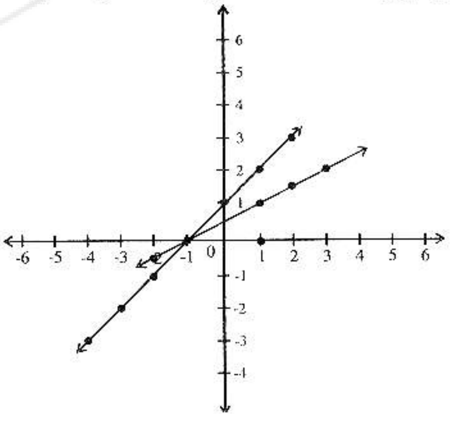
$2y = 1 + x$

$y = \dfrac{1+x}{2}$

Table of values,

x	−2	−1	0	1	2	3
y	−0.5	0	0.5	1	1.5	2

By plotting the points we get the following graph



The solution of the system is the point R where the lines ℓ and ℓ' meet at R $(-1, 0)$ such that $x = -1$ and $y = 0$

Q.3 $2x + y = 0$ and $x + 2y = 2$

Solution: $y = -2x$

Table of the values

x	−2	−1	0	1	2	3	4
y	4	2	0	−2	−4	−6	−4

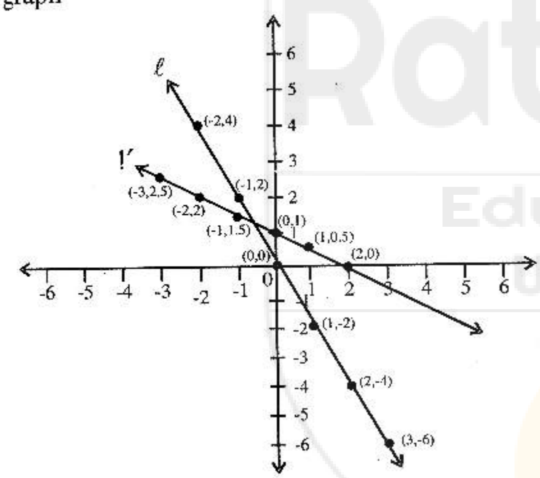
$x + 2y = 2$

$2y = 2 - x$

$y = \dfrac{2-x}{2}$

x	−3	−2	−1	0	1	2
y	2.5	2	1.5	1	0.5	0

By plotting the points we get the following graph



The solution of equations is $R\left(-\dfrac{2}{3}, \dfrac{4}{3}\right)$

Q.4 $x + y - 1 = 0$
$x - y + 1 = 0$

Solution: $x + y = 1$

$y = 1 - x$

Table of values

x	−3	−2	−1	0	1	2
y	4	3	2	1	0	−1

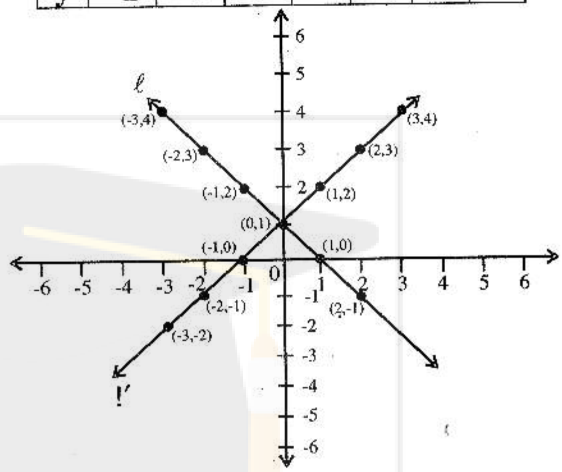
$x - y + 1 = 0$

$-y = -1 - x$

$y = 1 + x$

Table of values,

x	−3	−2	−1	0	1	2	3
y	−2	−1	0	1	2	3	4



The solution of the systems is R(0,1)

Q.5 $2x + y - 1 = 0,\ x = -y$

Solution: $2x + y = 1$

$y = 1 - 2x$

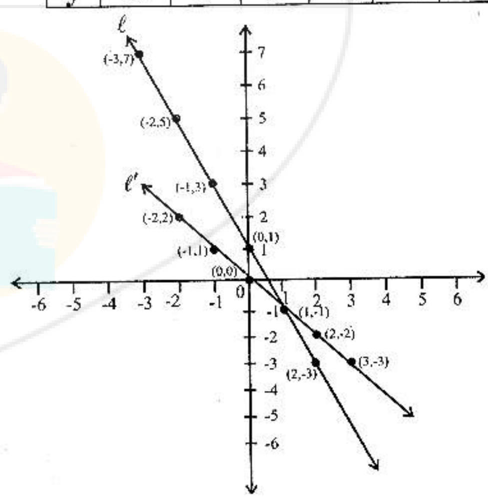
Table of values

x	−3	−2	−1	0	1	2
y	7	5	3	1	−1	−3

$x = -y$

Table of values

x	−2	−1	0	1	2	3
y	2	1	0	−1	−2	−3



The solution of the system is the point R where the lines ℓ and ℓ' meet at $R(1, -1)$ such that $x = 1$ and $y = -1$.

Objective

1. If $(x-1, y+1) = (0, 0)$, then (x, y) is:
(a) $(1, -1)$ (b) $(-1, 1)$
(c) $(1, 1)$ (d) $(-1, -1)$

2. If $(x, 0) = (0, y)$, then (x, y) is:
(a) $(0, 1)$ (b) $(1, 0)$
(c) $(0, 0)$ (d) $(1, 1)$

3. Point $(2\ -3)$ lies in quadrant:
(a) I (b) II
(c) III (d) IV

4. Point $(-3, -3)$ lies in quadrant:
(a) I (b) II
(c) III (d) IV

5. If $y = 2x + 1$, $x = 2$ then y is:
(a) 2 (b) 3
(c) 4 (d) 5

6. Which ordered pair satisfy the equation $y = 2x$:
(a) $(1, 2)$ (b) $(2, 1)$
(c) $(2, 2)$ (d) $(0, 1)$

7. The real numbers x, y of the ordered pair (x, y) are called ______ of point P(x,y) in a plane:
(a) co-ordinates
(b) x co-ordinates
(c) y-coordinate
(d) ordinate

8. Cartesian plane is divided into ___ quadrants:
(a) Two (b) Three
(c) Four (d) Five

9. The point of intersection of two coordinate axes is called:
(a) Origin (b) Centre
(c) X-coordinate (d) y-coordinate

10. The x-coordinate of a point is called__
(a) Origin (b) abcissa
(c) y-coordinate (d) Ordinate

11. The y-coordinate of a point is called:
(a) Origin (b) x-coordinate
(c) y-coordinate (d) ordinate

12. The set of points which lie on the same line are called ___ points:
(a) Collinear (b) Similar
(c) Common (d) None of these

13. The plane formed by two straight lines perpendicular to each other is called ___:
(a) Cartesian plane
(b) Coordinate axes
(c) Plane
(d) None of these

14. An ordered pair is a pair of elements in which elements are written in specific:
(a) Order (b) Array
(c) Point (d) None

Answer key

1.	a	**2.**	c	**3.**	d	**4.**	c	**5.**	d
6.	a	**7.**	a	**8.**	c	**9.**	a	**10.**	b
11.	d	**12.**	a	**13.**	a	**14.**	a		

Unit 09

INTRODUCTION TO COORDINATE GEOMETRY

Define Coordinate Geometry

The study of geometrical shapes in a plane is called plane geometry. Coordinate geometry is the study of geometrical shapes in the Cartesian plane (coordinate plane).

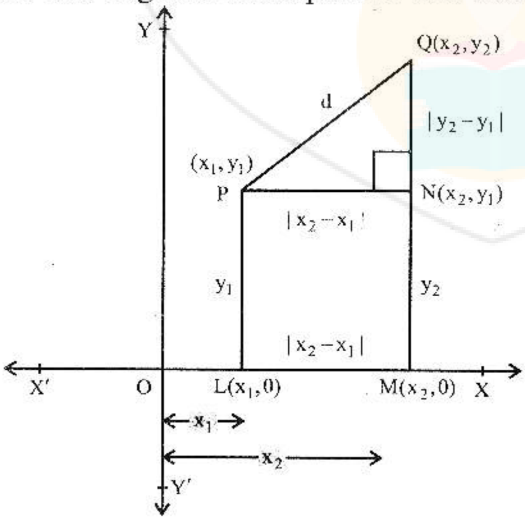
We known that a plane is divided into four quadrants by two perpendicular lines called the axes intersecting at origin. We have also seen that there is one to one correspondence between the points of the plane and the ordered pairs in $R \times R$.

Finding Distance between two points

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points in the coordinate plane where d is the length of the line segment PQ. i.e. $|PQ| = d$.

The line segments MQ and LP parallel to y–axis meet x-axis at points M and L, respectively with coordinates $M(x_2, 0)$ and $L(x_1, 0)$.

The line–segment PN is parallel to x–axis.



In the right triangle PNQ,

$|QN| = |y_2 - y_1|$ and $|PN| = |x_2 - x_1|$.

Using Pythagoras Theorem

$$\left(\overline{PQ}\right)^2 = \left(\overline{PN}\right)^2 = \left(\overline{QN}\right)^2$$

$$\Rightarrow \quad d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$\Rightarrow \quad d = \pm\sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$

$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$

since $d > 0$ always.

Example

Using the distance formula, find the distance between the points.

(i) P(1, 2) and Q(0,3)

(ii) S(–1, 3) and R(3, –2)

Solutions

(i) $|PQ| = \sqrt{(0-1)^2 + (3-2)^2}$

$= \sqrt{(-1)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$

(ii) $|SR| = \sqrt{(3-(-1))^2 + (-2-3)^2}$

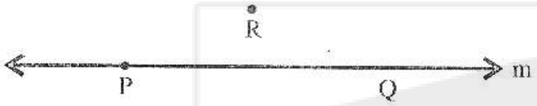
$= \sqrt{(3+1)^2 + (-5)^2} = \sqrt{16+25} = \sqrt{41}$

Collinear or Non-collinear Points in the Plane

Two or more than two points which lie on the same straight line are called collinear points with respect to that line; otherwise they are called non-collinear.

Let PQ be a line, then all the points on line m are collinear.

In the given figure the points P and Q are collinear with respect to the line m and the points P and R are not collinear with respect to it.



Use of Distance Formula to show the Collinearity of Three or more Points in the Plane

Let P, Q and R be three points in the plane. They are called collinear

If $|PQ| + |QR| = |PR|$, otherwise they are non-collinear.

Example

Using distance formula show that the points.

(i) $P(-2,-1)$, $Q(0, 3)$ and $R(1, 5)$ are collinear.

(ii) The above P,Q,R and S $(1,-1)$ are not collinear

Sol. By using the distance formula, we find

$$|PQ| = \sqrt{(0+2)^2 + (3+1)^2}$$
$$= \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$$
$$|QR| = \sqrt{(1-0)^2 + (+5-3)^2}$$
$$= \sqrt{1+4} = \sqrt{5}$$
$$|PR| = \sqrt{(1+2)^2 + (5+1)^2}$$
$$= \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$$

Since $|PQ| + |QR| = 2\sqrt{5} + \sqrt{5}$
$$= 3\sqrt{5} = |PR|$$

points P, Q, R are collinear.

(ii) The above points P,Q,R and S (1,-1) are not collinear

Sol $|PS| = \sqrt{(-2-1)^2 + (-1+1)^2}$
$$= \sqrt{(-3)^2 + 0} = 3$$

Since $|QS| = \sqrt{(1-0)^2 + (-1-3)^2}$
$$= \sqrt{1+16} = \sqrt{17},$$

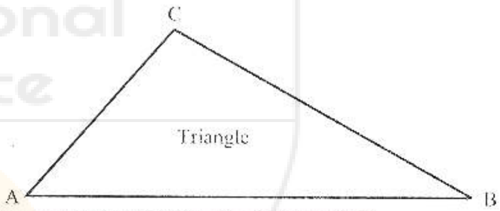
and $|PQ| + |QS| \neq |PS|$,

Therefore the points, P,Q and S are not collinear and hence, the points P, Q, R and S are also not collinear.

Define Triangle

A closed figure in a plane obtained by joining three non-collinear points is called a triangle.

In the triangle ABC the non-collinear points A, B and C are the three vertices of the triangle ABC. The line segments AB, BC and CA are called sides of the triangle.



Define Equilateral Triangle

If the lengths of all the three sides of a triangle are same, then the triangle is called an equilateral triangle.

Example

The triangle OPQ is an equilateral triangle since the points $O(0,0)$, $P\left(\frac{1}{\sqrt{2}}, 0\right)$ and $Q\left(\frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}\right)$ are not collinear, where

$$|OP| = \frac{1}{\sqrt{2}}$$

$$|QO| = \sqrt{\left(0-\frac{1}{2\sqrt{2}}\right)^2+\left(0-\frac{\sqrt{3}}{2\sqrt{2}}\right)^2}$$

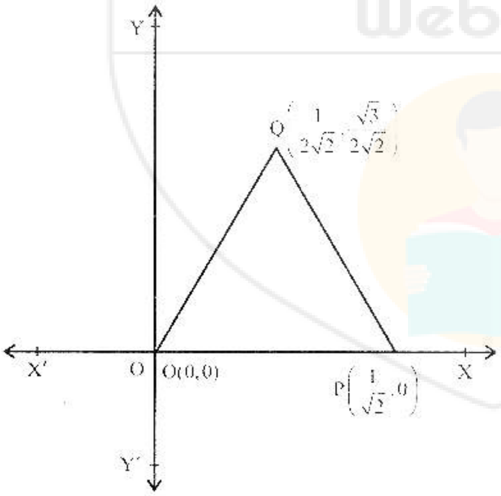
$$=\sqrt{\frac{1}{8}+\frac{3}{8}}=\sqrt{\frac{4}{8}}=\sqrt{\frac{1}{2}}$$

$$|PQ|=\sqrt{\left(\frac{1}{2\sqrt{2}}-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{\sqrt{3}}{2\sqrt{2}}-0\right)^2}$$

$$=\sqrt{\left(\frac{1-2}{2\sqrt{2}}\right)^2+\left(\frac{\sqrt{3}}{2\sqrt{2}}\right)^2}$$

$$=\sqrt{\frac{1}{8}+\frac{3}{8}}=\sqrt{\frac{4}{8}}=\sqrt{\frac{1}{2}}$$

i.e., $|OP|=|QO|=|PQ|=\frac{1}{\sqrt{2}}$, a real number and the points $O(0,0)$, $Q\left(\frac{1}{2\sqrt{2}},\frac{\sqrt{3}}{2\sqrt{2}}\right)$ and $P\left(\frac{1}{\sqrt{2}},0\right)$ are not collinear. Hence the triangle OPQ is equilateral.

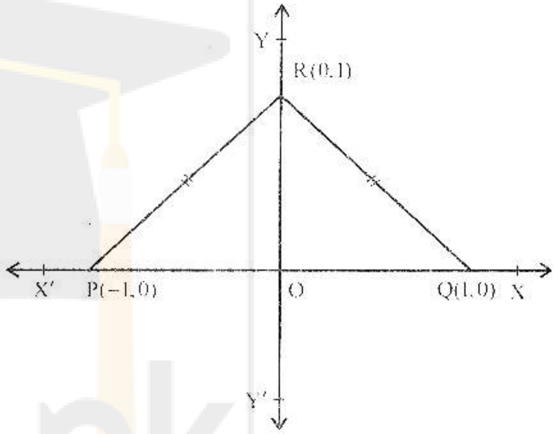


An Isosceles Triangle

An isosceles triangle PQR is a triangle which has two of its sides with equal length while the third side has a different length.

Example

The triangle PQR is an isosceles triangle as for the non-collinear points $P(-1,0)$, $Q(1, 0)$ and $R(0, 1)$ shown in the following figure.



$$|PQ|=\sqrt{(1-(-1))^2+(0-0)^2}=\sqrt{(1+1)^2+0}=\sqrt{4}=2$$

$$|QR|=\sqrt{(0-1)^2+(1-0)^2}=\sqrt{(-1)^2+1^2}=\sqrt{1+1}=\sqrt{2}$$

$$|PR|=\sqrt{(0-(-1))^2+(1-0)^2}=\sqrt{1+1}=\sqrt{2}$$

Since $|QR|=|PR|=\sqrt{2}$ and $|PQ| = 2\neq \sqrt{2}$ so the non-collinear points P, Q, R form an isosceles triangle PQR.

Right Angle Triangle

A triangle in which one of the angles has measure equal to 90° is called a right angle triangle.

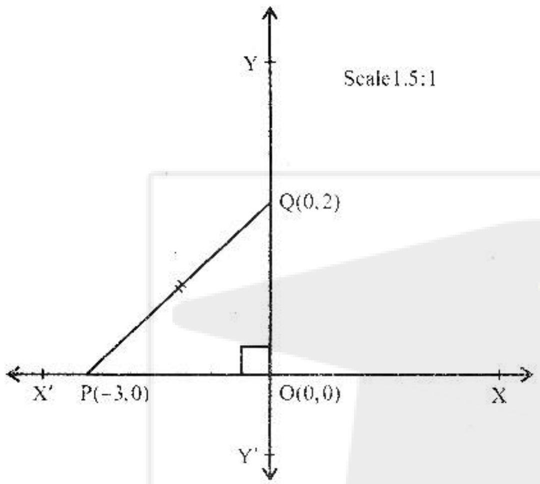
Example

Let $O(0, 0)$, $P(-3, 0)$ and $Q(0, 2)$ be three non-collinear points. Verify that triangle OPQ is right-angled.

$$|OQ|=\sqrt{(0-0)^2+(2-0)^2}=\sqrt{2^2}=2$$

$$|OP|=\sqrt{(-3)^2+0^2}=\sqrt{9}=3$$

$$|PQ| = \sqrt{(-3)^2+(-2)^2}=\sqrt{9+4}=\sqrt{13}$$



Now

$|OQ|^2 + |OP|^2 = (2)^2 + (3)^2 = 13$ and $|PQ|^2 = 13$

Since

$|OQ|^2 + |OP|^2 = |PQ|^2$, therefore $\angle POQ = 90^\circ$

Hence the given non-collinear points form a right triangle.

Scalene Triangle

A triangle is called a scalene triangle if measures of all the three sides are different.

Example

Show that the points P(1, 2), Q(−2, 1) and R(2, 1) in the plane form a scalene triangle.

Solution

$|PQ| = \sqrt{(-2-1)^2 + (1-2)^2}$

$= \sqrt{(-3)^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10}$

$|QR| = \sqrt{(2+2)^2 + (1-1)^2}$

$= \sqrt{4^2 + 0^2} \quad = \sqrt{4^2} = 4$

and

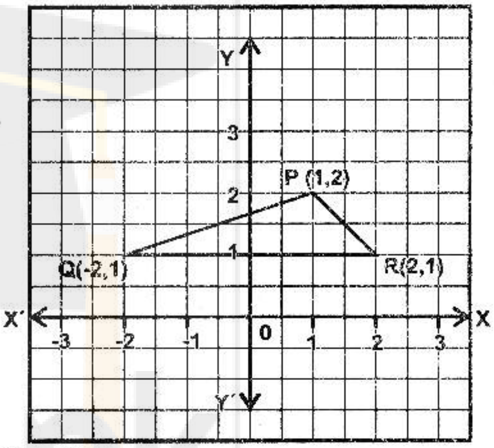
$|PR| = \sqrt{(2-1)^2 + (1-2)^2}$

$= \sqrt{1^2 + (-1)^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

Hence $|PQ| = \sqrt{10}, |QR| = 4$ and $|PR| = \sqrt{2}$

The points P, Q and R are non-collinear since, $|PQ| + |QR| > |PR|$

Thus the given points form a scalene triangle.



Example

If A(2, 2), B(2, −2), C(−2, −2) and D(−2, 2) be four non-collinear points in the plane, then verify that they form a square ABCD.

Solution

Since $|AB| = \sqrt{(2-2)^2 + (-2-2)^2}$

$= \sqrt{0^2 + (-4)^2} = \sqrt{16} = 4$

$|BC| = \sqrt{(-2-2)^2 + (-2+2)^2}$

$= \sqrt{(-4)^2 + 0^2} = \sqrt{16} = 4$

$|CD| = \sqrt{(-2-(-2))^2 + (2-(-2))^2}$

$= \sqrt{(-2+2)^2 + (2+2)^2}$

$= \sqrt{0+16} = \sqrt{16} = 4$

$|DA| = \sqrt{(2+2)^2 + (2-2)^2}$

$= \sqrt{(+4)^2 + 0} = \sqrt{16} = 4$

Hence $|AB| = |BC| = |CD| = |DA| = 4$

Also $|AC|=\sqrt{(-2-2)^2+(-2-2)^2}$

$=\sqrt{16+16}=\sqrt{32}=4\sqrt{2}$

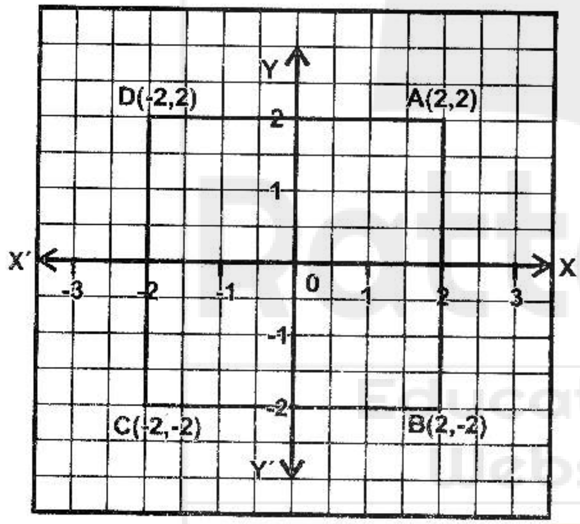
Now $|AB|^2+|BC|^2=(4)^2+(4)^2=32$, and

$|AC|^2=(4\sqrt{2})^2=32$

Since $|AB|^2+|BC|^2=|AC|^2$,

therefore $\angle ABC=90^\circ$

Hence the given four non-collinear points form a square.



Define Parallelogram

A figure formed by four non-collinear points in the plane is called a parallelogram if

(i) its opposite sides are of equal length

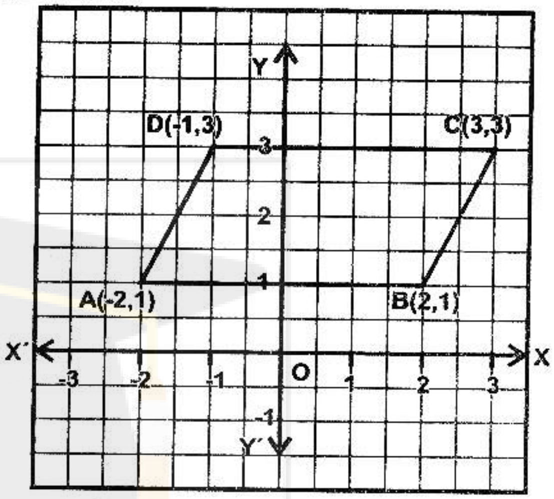
(ii) its opposite sides are parallel

(iii) measure of none of the angles in 90°

Example

Show that the points A(−2, 1), B(2, 1), C(3, 3) and D(−1, 3) form a parallelogram.

Solution:



By distance formula,

$|AB|=\sqrt{(2+2)^2+(1-1)^2}$

$=\sqrt{4^2+0}=\sqrt{16}=\quad 4$

$|CD|=\sqrt{(3+1)^2+(3-3)^2}$

$=\sqrt{4^2+0}=\sqrt{16}=4$

$|AD|=\sqrt{(-1+2)^2+(3-1)^2}$

$=\sqrt{1^2+2^2}=\sqrt{1+4}=\sqrt{5}$

$|BC|=\sqrt{(3-2)^2+(3-1)^2}$

$=\sqrt{1^2+2^2}=\sqrt{5}$

Since

$|AB|=|CD|=4$ and $|AD|=|BC|=\sqrt{5}$

So opposite sides of the quadrilateral ABCD are equal.

Also $|AC|=\sqrt{(3+2)^2+(3-1)^2}$

$=\sqrt{(5)^2+2^2}=\sqrt{25+4}=\sqrt{29}$

Now

$|AB|^2+|BC|^2=16+5=21$ and $|AC|^2=29$

Since in triangle

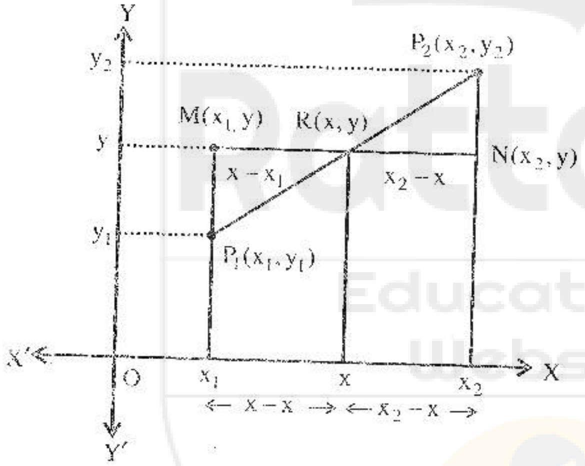
$ABC, |AB|^2 + |BC|^2 \neq |AC|^2$

Therefore measure of angle at $B \neq 90^o$

Hence the given points form a parallelogram.

Recognition of the Mid-Point Formula for any two Points in the Plane

Let $P_1(x_1, y_1)$ and $P_2(x_1, y_1)$ be any two points in the plane and $R(x, y)$ be a mid-point of points P_1 and P_2 on the line-segment P_1P_2 as shown in the figure below.



If line-segment MN, parallel to x-axis, has its mid-point $R(x,y)$,

then , $x_2 - x = x - x_1$

$$\Rightarrow \quad 2x = x_1 + x_2 \Rightarrow x = \frac{x_1 + x_2}{2}$$

Similarly, $y = \frac{y_1 + y_2}{2}$

Thus the point $R(x, y) = R\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ is the mid-point of the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Example

Find the mid-point of the line segment joining A(2,5) and B(−1, 1).

Solution

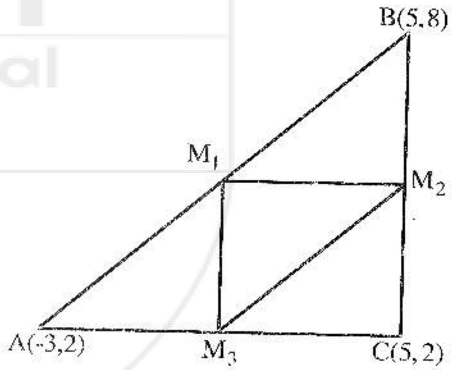
If $R(x, y)$ is the desired mid–point then.

$$x = \frac{2-1}{2} = \frac{1}{2} \text{ and } y = \frac{5+1}{2} = \frac{6}{2} = 3$$

Hence $R(x, y) = R\left(\frac{1}{2}, 3\right)$

xample

Let ABC be a triangle as shown below. If M_1, M_2 and M_3 are the middle points of the line-segments AB, BC and CA respectively, find the coordinates of M_1, M_2 and M_3. Also determine the type of the triangle $M_1M_2M_2$.



Solution

Midpoint of $AB = M_1\left(\frac{-3+5}{2}, \frac{2+8}{2}\right) = M_1(1,5)$

Midpoint of $BC = M_2\left(\frac{5+5}{2}, \frac{8+2}{2}\right) = M_2(5,5)$

and Mid–point of

$AC = M_3\left(\frac{5-3}{2}, \frac{2+2}{2}\right) = M_3(1,2)$

The triangle $M_1M_2M_3$ has sides with length,

$$|M_1M_2| = \sqrt{(5-1)^2+(5-5)^2}$$

$$= \sqrt{4^2+0} = 4 \;.....(i)$$

$$|M_2M_3| = \sqrt{(1-5)^2+(2-5)^2}$$

$$= \sqrt{(-4)^2+(-3)^2}$$

$$= \sqrt{16+9} = \sqrt{25} = 5 \quad .....(ii)$$

and $$|M_1M_3| = \sqrt{(1-1)^2+(2-5)^2}$$

$$= \sqrt{0^2+(-3)^2} = 3 .....(iii)$$

All the lengths of the three sides are different. Hence the triangle $M_1M_2M_3$ is a Scalene triangle.

Example

Let O(0,0), A(3,0) and B(3,5) be three points in the plane. If M_1 is the mid point of AB and M_2 of OB, then show that $|M_1M_2| = \frac{1}{2}|OA|$.

Solution

By the distance formula the distance

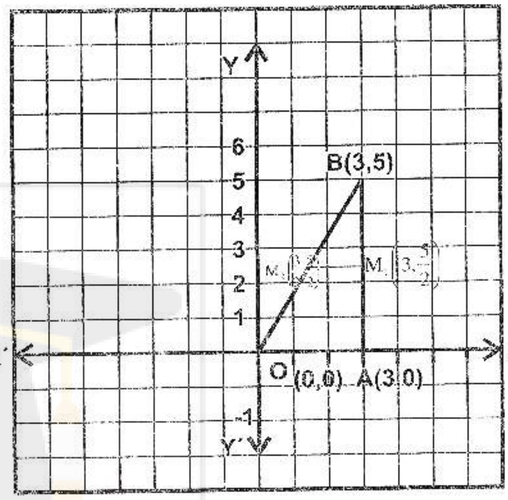
$$|OA| = \sqrt{(3-0)^2+(0-0)^2} = \sqrt{3^2} = 3$$

The mid–point of AB is:

$$M_1 = M_1\left(\frac{3+3}{2}, \frac{5+0}{2}\right) = \left(3, \frac{5}{2}\right)$$

The mid–point of OB is:

$$M_2 = M_2\left(\frac{3+0}{2}, \frac{5+0}{2}\right) = \left(\frac{3}{2}, \frac{5}{2}\right)$$



Hence

$$|M_1M_2| = \sqrt{\left(\frac{3}{2}-3\right)^2+\left(\frac{5}{2}-\frac{5}{2}\right)^2}$$

$$= \sqrt{\left(\frac{-3}{2}\right)^2+0} = \sqrt{\frac{9}{4}+0} = \frac{3}{2}$$

$$= \frac{1}{2}|OA|$$

Note:

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be any two points and their midpoint be:

$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. Then M

(i) is at equal distance from P and Q i.e., $|PM| = |MQ|$

(ii) is an interior point of the line segment PQ.

(iii) every point R in the plane at equal distance from P and Q is not their mid point. For example, the point R(0,1) is at equal distance from P(–3, 0) and Q(3, 0) but is not their mid–point.

i.e., $|RQ| = \sqrt{(0-3)^2+(1-0)^2}$

$= \sqrt{(-3)^2 + (1)^2} = \sqrt{9+1} = \sqrt{10}$

$|RP| = \sqrt{(0+3)^2 + (1-0)^2}$

$= \sqrt{3^2 + 1^2} = \sqrt{10}$

And midpoint of P(–3,0) and Q(3, 0) is

Where $x = \dfrac{-3+3}{2} = 0$

$$y = \frac{0+0}{2} = 0$$

The point (0, 1) $\neq$ (0, 0).

(iv) There is a unique midpoint of any two points in the plane.

Exercise 9.1

Q1. Find the distance between the following pairs of points

a) $A(9, 2),\ B(7,2)$

Sol. $|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2}$

$= \sqrt{(7-9)^2 + (2-2)^2}$

$= \sqrt{(-2)^2 + (0)^2}$

$= \sqrt{4}$

$= 2$

b) $A(2,-6),\ B(3,-6)$

Sol. $|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(3-2)^2 + (-6+6)^2}$

$= \sqrt{(1)^2 + (0)^2}$

$= \sqrt{1}$

$= 1$

c) $A(-8, 1),\ B(6, 1)$

Sol. $|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(6+8)^2 + (1-1)^2}$

$= \sqrt{(14)^2 + (0)^2}$

$|AB| = 14$

d) $A(-4, \sqrt{2}),\ B(-4, -3)$

Sol. $|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y)^2}$

$= \sqrt{(-4+4)^2 + (-3-\sqrt{2})^2}$

$= \sqrt{(0)^2 + (-3-\sqrt{2})^2}$

$= \sqrt{(-3-\sqrt{2})^2}$

$= (3+\sqrt{2})^2$

$= 3+\sqrt{2}$

(e) $A(3,-11),\ B(3,-4)$

Sol. $|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(3-3)^2 + (-4-(-11))^2}$

$= \sqrt{(0)^2 + (7)^2} = \sqrt{(7)^2} = 7$

(f) A(0,0) B(0,-5)

$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Sol. $= \sqrt{(0-0)^2 + (-5-0)^2}$

$= \sqrt{0 + (-5)^2} = \sqrt{(5)^2} = 5$

Q2. Let P be the point on x-axis with x-coordinate a and Q be the point on y-axis with y-coordinate b, as given below. Find distance between P and Q.

i) $a = 9$, $b = 7$

$|PQ| = \sqrt{(9)^2 + (7)^2} = \sqrt{81+49} = \sqrt{130}$

ii) $a = 2$, $b = 3$

$|PQ| = \sqrt{(2)^2 + (3)^2} = \sqrt{4+9} = \sqrt{13}$

iii) $a = -8$, $b = 6$

$|PQ| = \sqrt{(-8)^2 + (6)^2} = \sqrt{64+36} = \sqrt{100} = 10$

iv) $a = -2$, $b = -3$

$|PQ| = \sqrt{(-2)^2 + (-3)^2} = \sqrt{4+9} = \sqrt{13}$

v) $a = \sqrt{2}$, $b = 1$

$|PQ| = \sqrt{\left(\sqrt{2}\right)^2 + (1)^2} = \sqrt{2+1} = \sqrt{3}$

vi) $a = -9$, $b = -4$

$|PQ| = \sqrt{(-9)^2 + (-4)^2} = \sqrt{81+16} = \sqrt{97}$

Exercise 9.2

Q1. Show whether the points with vertices $(5,-2)$, $(5,4)$ and $(-4,1)$ are vertices of an equilateral triangle or an isosceles triangle?

SOL. Let P(5,-2) , Q(5,4), R(-4,1)

$|PQ| = \sqrt{(5-5)^2 + (4+2)^2} = \sqrt{(0)^2 + (6)^2} = \sqrt{36} = 6$

$|QR| = \sqrt{(-4-5)^2 + (1-4)^2} = \sqrt{81+9} = \sqrt{90}$

$|PR| = \sqrt{(-4-5)^2 + (1+2)^2} = \sqrt{81+9} = \sqrt{90}$

Since $|QR| = |PR| = \sqrt{90}$ and

$|PQ| = 6 \neq \sqrt{90}$

So the non collinear points P, Q, R form an isosceles triangle PQR

Q2. Show whether or not the points with vertices $(-1,1)$, $(5,4)$, $(2,-2)$ and $(-4,1)$ form a square.

Sol. Let $A(-1,1)$, $B(5,4)$, $C(2,-2)$, $D(-4,1)$

Since $|AB| = \sqrt{(5+1)^2 + (4-1)^2}$

$= \sqrt{6^2 + 3^2} = \sqrt{36+9} = \sqrt{45}$

$|BC| = \sqrt{(2-5)^2 + (-2-4)^2}$

$|BC| = \sqrt{(-3)^2 + (-6)^2} = \sqrt{9+36} = \sqrt{45}$

$|CD| = \sqrt{(-4-2)^2 + (1+2)^2}$

$= \sqrt{(-6)^2 + (3)^2} = \sqrt{36+9} = \sqrt{45}$

$|DA| = \sqrt{(-4+1)^2 + (1-1)^2}$

$= \sqrt{(-3)^2 + (0)^2} = \sqrt{9} = 3$

Hence $|AB| = |BC| = |CD| = \sqrt{45}$

but $|DA| \neq \sqrt{45}$

Hence given points do not form a square.

Q3. Show whether or not the points with coordinates $(1,3)(4,2)$, and $(-2,6)$ are vertices of a right triangle.

Sol. Let $P(1,3)$, $Q(4,2)$ and $R(-2,6)$

$|PQ| = \sqrt{(4-1)^2 + (2-3)^2}$

$= \sqrt{(3)^2 + (1)^2} = \sqrt{9+1} = \sqrt{10}$

$$|QR|=\sqrt{(-2-4)^2+(6-2)^2}$$
$$=\sqrt{36+16}=\sqrt{52}$$
$$|PR|=\sqrt{(-2-1)^2+(6-3)^2}$$
$$|BC|=\sqrt{(-3)^2+(3)^2}=\sqrt{9+9}=\sqrt{18}=3\sqrt{2}$$

Now $|PQ|^2+|QR|^2=\left(\sqrt{10}\right)^2+\left(\sqrt{52}\right)^2$

$$=10+52=62$$

and $|PR|^2=\left(\sqrt{18}\right)^2=18$

$$|PQ|^2+|QR|^2\neq|PR|^2$$

So triangle is not right angled

Q4. Use the distance formula to prove whether or not the points $(1,1),(-2,-8)$ and $(4,10)$ lie on a straight line.

Let $A(1,1)$, $B(-2,-8)$, $C(4,10)$

Since $|AB|=\sqrt{(-2-1)^2+(-8-1)^2}$

$$=\sqrt{(-3)^2+(-9)^2}=\sqrt{9+81}=\sqrt{90}=3\sqrt{10}$$
$$|BC|=\sqrt{(4+2)^2+(10+8)^2}$$
$$|BC|=\sqrt{(6)^2+(18)^2}$$
$$=\sqrt{36+324}=\sqrt{360}$$
$$=\sqrt{2\times2\times2\times3\times3\times5}=6\sqrt{10}$$
$$|AC|=\sqrt{(4-1)^2+(10-1)^2}$$
$$=\sqrt{(3)^2+(9)^2}=\sqrt{9+81}=\sqrt{90}=3\sqrt{10}$$
$$|AB|+|AC|=3\sqrt{10}+3\sqrt{10}$$
$$=6\sqrt{10}=|BC|$$

So $|AB|+|AC|=|BC|$ the points A, B and C are collinear.

Q5. Find K given that the point $(2,K)$ is equidistance from $(3,7)$ and $(9,1)$.

Sol. Let $P(2,K), Q(3,7)$ and $R(9,1)$

$$|PQ|=\sqrt{(3-2)^2+(7-K)^2}$$
$$=\sqrt{1^2+(7-K)^2}=\sqrt{1+(7-K)^2}$$
$$=\sqrt{1+49-2(7)k+k^2}$$
$$=\sqrt{50-14k+k^2}$$
$$|PR|=\sqrt{(9-2)^2+(1-K)^2}$$
$$=\sqrt{49+1-2(1)k+k^2}$$
$$=\sqrt{50-2k+k^2}$$

As point P is equidistant from Q and

So $|PQ|=|PR|$

$$\sqrt{50-14k+k^2}=\sqrt{50-2k+k^2}$$
$$50-14k+k^2=50-2k+k^2$$
$$-12k=0\Rightarrow k=0$$

Q6. Use distance formula to verify that the points $A(0,7)$, $B(3,-5)$, $C(-2,15)$ are collinear.

So $|AB|=\sqrt{(3-0)^2+(-5-7)^2}$

$$=\sqrt{9+(-12)^2}=\sqrt{9+144}$$
$$=\sqrt{153}=12.37$$
$$|BC|=\sqrt{(-2-3)^2+(15+5)^2}$$
$$=\sqrt{25+400}=\sqrt{425}=20.62$$
$$|CA|=\sqrt{(-2-0)^2+(15-7)^2}$$
$$=\sqrt{4+64}=\sqrt{68}=8.25$$

As $|AB|+|CA|=|BC|$

So given points are collinear with A between B and C.

Q7. Verify whether or not the points $O(0,0)$, $A\left(\sqrt{3},1\right)$, $B\left(\sqrt{3}-1\right)$ are vertices of a equilateral triangle.

Sol. $|OA|=\sqrt{(\sqrt{3}-0)+(1-0)^2}$

$=\sqrt{\left(\sqrt{3}\right)^2+(1)^2}$

$=\sqrt{3+1}=\sqrt{4}=2$

$|AB|=\sqrt{\left(\sqrt{3}-\sqrt{3}\right)^2+(-1-1)^2}$

$=\sqrt{(0)^2+(-2)^2}=\sqrt{0+4}=2$

$|OB|=\sqrt{\left(\sqrt{3}-0\right)^2+(-1-0)^2}$

$=\sqrt{\left(\sqrt{3}\right)^2+(-1)^2}$

$=\sqrt{3+1}=\sqrt{4}=2$

As $|OA|=|AB|=|OB|=2$

Hence points are not collinear.

$\therefore$ the triangle OAB is equilateral

Q8. Show that the points A(-6,-5), B(5,-5), C(5,-8), D(-6,-8) are vertices of a rectangle. Find the lengths of its diagonals. Are they equal ?.

Sol. $|AB|=\sqrt{(5+6)^2+(-5+5)^2}$

$=\sqrt{(11)^2+(0)^2}=\sqrt{121}=11$

$|BC|=\sqrt{(5-5)^2+(-8+5)^2}$

$=\sqrt{(0)^2+(-3)^2}=\sqrt{9}=3$

$|DC|=\sqrt{(5+6)^2+(-8+8)^2}$

$=\sqrt{(11)^2+(0)^2}=\sqrt{121}=11$

$|AD|=\sqrt{(-6+6)^2+(-8+5)^2}$

$=\sqrt{(-3)^2}=\sqrt{9}=3$

Since $|AB|=|DC|=11$ and $|AD|=|BC|=3$ opposite sides are equal

Diagonal $|AC|=\sqrt{(5+6)^2+(-8+5)^2}$

$=\sqrt{11^2+3^2}=\sqrt{121+9}=\sqrt{130}$

Diagonal $|BD|=\sqrt{(-6-5)^2+(-8+5)^2}$

$=\sqrt{11^2+3^2}=\sqrt{121+9}=\sqrt{130}$

$|AD|^2+|DC|^2=|AC|^2$

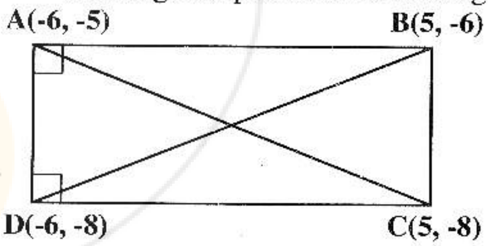
$\therefore \quad \angle ADC=90^o$

Also $|AB|^2+|AD|^2=|BD|^2$

$\therefore \angle BAD=90^o$

$|AC|=|BD|=\sqrt{130}$

Hence given points form rectangle



As $|AC|=|BD|=\sqrt{130}$

Hence diagonals are equal.

Q9. Show that the points $M(-1,4)$, $N(-5,3)$, $P(1,-3)$ and $Q(5,-2)$ are the vertices of a parallelogram.

SOL. $|PQ|=\sqrt{(5-1)^2+(-2+3)^2}$

$=\sqrt{(4)^2+(1)^2}=\sqrt{16+1}=\sqrt{17}$

$$|MN|=\sqrt{(-5+1)^2+(3-4)^2}$$
$$=\sqrt{(-4)^2+(-1)^2}=\sqrt{16+1}=\sqrt{17}$$
$$|NP|=\sqrt{(1+5)^2+(-3-3)^2}$$
$$=\sqrt{(6)^2+(-6)^2}=\sqrt{36+36}=\sqrt{72}$$
$$|MQ|=\sqrt{(5+1)^2+(-2-4)^2}$$
$$=\sqrt{6^2+(-6)^2}=\sqrt{36+36}=\sqrt{72}$$

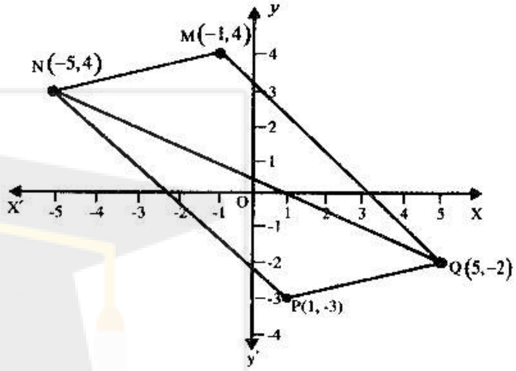
Since $|PQ|=|MN|=\sqrt{17}$

and $|NP|=|MQ|=\sqrt{72}$

So opposite sides, of quadrilateral MNPQ are equal.

$$|NQ|=\sqrt{(-5-5)^2+(3+2)^2}$$
$$=\sqrt{(-10)^2+(5)^2}$$
$$=\sqrt{100+25}=\sqrt{125}=5\sqrt{5}$$
$$|PN|^2+|PQ|^2=\left(\sqrt{72}\right)^2+\left(\sqrt{17}\right)^2$$
$$=72+17=89$$
$$|PN|^2+|PQ|^2\neq|NQ|^2$$

The measure of angle at $P\neq 90^o$



Hence given points form a parallelogram.

Q10. Find the length of the diameter of the circle having centre at $C(-3,6)$ and passing through $P(1,3)$.

SOL. Length of radius=

$$|PC|=\sqrt{(-3-1)^2+(6-3)^2}$$
$$=\sqrt{(-4)^2+(3)^2}$$
$$=\sqrt{16+9}$$
$$=\sqrt{25}$$
$$=5$$

Length of diameter $=2r=2(r)=10$

Exercise 9.3

Q1. Find the mid point of the line segment joining each of the following pairs of points.

a) $A(9,2)$, $B(7,2)$

If $R(x,y)$ is the desired midpoint then,

$$x=\frac{x_1+x_2}{2}=\frac{9+7}{2}=\frac{16}{2}=8$$
$$y=\frac{y_1+y_2}{2}=\frac{2+2}{2}=\frac{4}{2}=2$$
$$\therefore\ R(x,y)=R(8,2)$$

b) $A(2,6)$, $B(3,-6)$

If $R(x,y)$ is the desired midpoint then,

$$x=\frac{x_1+x_2}{2}=\frac{2+3}{2}=\frac{5}{2}$$
$$y=\frac{y_1+y_2}{2}=\frac{6-6}{2}=\frac{0}{2}=0$$
$$R(x,y)=R\left(\frac{5}{2},0\right)$$

c) $A(-8,1)$, $B(6,1)$

If $R(x,y)$ is the desired midpoint then

$$x=\frac{x_1+x_2}{2}=\frac{-8+6}{2}=\frac{-2}{2}=-1$$

$$y=\frac{y_1+y_2}{2}=\frac{1+1}{2}=\frac{2}{2}=1$$

$$\therefore\ R(x,y)=R(-1,1)$$

d) $A(-4,9)$, $B(-4,-3)$

If $R(x,y)$ is the desired mid point then,

$$x=\frac{x_1+x_2}{2}=\frac{-4-4}{2}=\frac{-8}{2}=-4$$

$$y=\frac{y_1+y_2}{2}=\frac{9-3}{2}=\frac{6}{2}=3$$

$$R(x,y)=R(-4,3)$$

e) $A(3,-11)$, $B(3,-4)$

If $R(x,y)$ is the desired midpoint then,

$$x=\frac{x_1+x_2}{2}=\frac{3+3}{2}=\frac{6}{2}=3$$

$$y=\frac{y_1+y_2}{2}=\frac{-11-4}{2}=\frac{-15}{2}=-7.5$$

$$\therefore\ R(x,y)=R(3,-7.5)$$

f) $A(0,0)$, $B(0,-5)$

If $R(x,y)$ is the desired midpoint then,

$$x=\frac{x_1+x_2}{2}=\frac{0+0}{2}=0$$

$$y=\frac{y_1+y_2}{2}=\frac{0-5}{2}=\frac{-5}{2}=-2.5$$

$$\therefore\ R(x,y)=R(0,-2.5)$$

Q2. The end point P of a line segment PQ (–3,6) and its mid point is (5,8). Find the co-ordinates of the end point Q.

Sol: $(-3,6)$

If $R(x,y)$ is mid point then,

$$x=\frac{x_1+x_2}{2}\Rightarrow 5=\frac{-3+x_2}{2}$$

$$\Rightarrow 10=-3+x_2$$

$$x_2=10+3=13$$

and $y=\frac{y_1+y_2}{2}\Rightarrow 8=\frac{6+y_2}{2}$

$$\Rightarrow 16=6+y_2$$

$$y_2=10$$

$\therefore$ Coordinates of the end point $Q(13,10)$

Q3. Prove that midpoint of the hypotenuse of a right triangle is equidistant from its three vertices $P(-2,5), Q(1,3)$ and $R(-1,0)$

SOL. $|PQ|^2=(1+2)^2+(3-5)^2=9+4=13$

$$|QR|^2=(-1-1)^2+(0-3)^2=4+9=13$$

$$|PR|^2=(-1+2)^2+(0-5)^2=1+25=26$$

As $|PQ|^2+|QR|^2=|PR|^2$

Hence PR is the hypotenuse

If $M(x,y)$ is desired midpoint then,

$$x=\frac{-1+(-2)}{2}=\frac{-1-2}{2}=\frac{-3}{2}$$

$$y=\frac{5+0}{2}=\frac{5}{2}$$

$$\therefore\quad M(x,y)=M\left(\frac{-3}{2},\frac{5}{2}\right)$$

Now $|PM|=\sqrt{\left(\frac{-3}{2}+2\right)^2+\left(\frac{5}{2}-5\right)^2}$

$$=\sqrt{\left(\frac{-3+4}{2}\right)^2+\left(\frac{5}{2}-10\right)^2}$$

$$=\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{-5}{2}\right)^2}$$

$$=\sqrt{\frac{1}{4}+\frac{25}{4}}$$

$$=\sqrt{\frac{26}{4}}$$

$$|RM|=\sqrt{\left(-\frac{3}{2}+1\right)^2+\left(\frac{5}{2}-0\right)^2}$$

$$=\sqrt{\left(\frac{-3+2}{2}\right)^2+\left(\frac{5-0}{2}\right)^2}$$

$$=\sqrt{\left(\frac{-1}{2}\right)^2+\left(\frac{5}{2}\right)^2}=\sqrt{\frac{1}{4}+\frac{25}{4}}$$

$$=\sqrt{\frac{1+25}{4}}=\sqrt{\frac{26}{4}}$$

$$|QM|=\sqrt{\left(-\frac{3}{2}-1\right)^2+\left(\frac{5}{2}-3\right)^2}$$

$$=\sqrt{\left(\frac{-3-2}{2}\right)^2+\left(\frac{5-6}{2}\right)^2}$$

$$=\sqrt{\left(\frac{-5}{2}\right)^2+\left(-\frac{1}{2}\right)^2}=\sqrt{\frac{25}{4}+\frac{1}{4}}==\sqrt{\frac{26}{4}}$$

As $|PM|=|RM|=|QM|$

$\therefore$ M is equidistant from P, Q and R.

Q4. O (0, 0), A(3, 0) and B(3, 5) are three points in the plane, find M_1 and M_2 as midpoints of the line segments $\overline{AB}$ and $\overline{OB}$ respectively. Find $|M_1, M_2|$.

Sol: Let O (0,0), A(3,0), B(3,5) are three points in the plane. M_1 is the mid point of $\overline{OB}$ and M_2 is the mid-point of $\overline{AB}$

$$M(x,y)=M\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

$$=M_1\left(\frac{0+3}{2},\frac{0+5}{2}\right)$$

$$=M_1\left(\frac{3}{2},\frac{5}{2}\right)$$

M_2 is midpoint of $\overline{AB}$ therefore

$$M_2\left(\frac{3+3}{2},\frac{0+5}{2}\right)=M_2\left(\frac{6}{2},\frac{5}{2}\right)$$

$$=M_2\left(3,\frac{5}{2}\right)$$

Now $\left(\frac{3}{2},\frac{5}{2}\right)$ and $\left(3,\frac{5}{2}\right)$ are midpoints

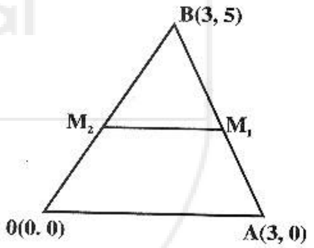
we find $|M_1 M_2|$

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Then $|M_1 M_2|=\sqrt{\left(3-\frac{3}{2}\right)^2+\left(\frac{5}{2}-\frac{5}{2}\right)}$

$$=\sqrt{\left(\frac{6-3}{2}\right)+0}$$

$$=\sqrt{\left(\frac{3}{2}\right)^2+0}=\frac{3}{2}$$



Q5. Show that the diagonals of the parallelogram having vertices $A(1,2)$, $B(4,2)$, $C(-1,-3)$, $D(-4,-3)$ bisect each other.

Sol: If M_1 is desired midpoint of diagonal DB.

$$x=\frac{x_1+x_2}{2}=\frac{4-4}{2}=0$$

$$y=\frac{y_1+y_2}{2}=\frac{2-3}{2}=\frac{-1}{2}$$

$$M_1(x,y)=\left(0,-\frac{1}{2}\right)$$

If M_2 is desired midpoint of diagonal AC

$$x=\frac{x_1+x_2}{2}=\frac{1-1}{2}=0$$

$$y=\frac{y_1+y_2}{2}=\frac{2-3}{2}=\frac{-1}{2}$$

$$M_2(x,y)=\left(0,-\frac{1}{2}\right)$$

∴ As midpoints of the diagonals coincide hence diagonal bisect each other.

Q6. The vertices of a triangle are P(4,6), Q(–2,–4) and R(–8, 2) show that the length of line segment joining the mid points of line segment PR, QR is $\frac{1}{2}$ PQ.

Sol. If M_1 is desired midpoint of line segment PR.

$$x=\frac{x_1+x_2}{2}=\frac{4-8}{2}=\frac{-4}{2}=-2$$

$$y=\frac{y_1+y_2}{2}=\frac{6+2}{2}=\frac{8}{2}=4$$

$$M_1(x,y)=M_1(-2,4)$$

If M_2 is desired midpoint of line segment QR.

$$x=\frac{x_1+x_2}{2}=\frac{-2-8}{2}=\frac{-10}{2}=-5$$

$$y=\frac{y_1+y_2}{2}=\frac{-4+2}{2}=\frac{-2}{2}=-1$$

$$M_2(x,y)=M_2(-5,-1)$$

$$|M_1M_2|=\sqrt{(-5+2)^2+(-1-4)^2}$$

$$=\sqrt{(-3)^2+(-5)^2}$$

$$=\sqrt{9+25}=\sqrt{34}$$

$$|PQ|=\sqrt{(-2-4)^2+(-4-6)^2}$$

$$=\sqrt{(-6)^2+(-10)^2}$$

$$=\sqrt{36+100}=\sqrt{136}=\sqrt{34\times4}$$

$$=2\sqrt{34}$$

As $2|M_1M_2|=|PQ|$

Hence $|M_1M_2|=\frac{1}{2}|PQ|$

Review Exercise 9

Q3. Find distance between pairs of points

i) $(6,3),(3,-3)$

Let $P(6,3),Q(3,-3)$

$$|PQ|=\sqrt{(3-6)^2+(-3-3)^2}$$

$$=\sqrt{(-3)^2+(-6)^2}$$

$$=\sqrt{9+36}=\sqrt{45}$$

ii) $(7,5),(1,-1)$

Let $P(7,5),Q(1,-1)$

$$|PQ|=\sqrt{(7-1)^2+(5+1)^2}$$

$$=\sqrt{(6)^2+(6)^2}=\sqrt{36+36}$$

$$=\sqrt{72}=\sqrt{36\times2}=6\sqrt{2}$$

iii) $(0,0),(-4,-3)$

Let $P(0,0),Q(-4,3)$

$$|PQ|=\sqrt{(-4-0)^2+(-3-0)^2}$$

$$=\sqrt{(-4)^2+(-3)^2}$$
$$=\sqrt{16+9}=\sqrt{25}=5$$

Q4. Find the midpoint between the following pairs of points.

SOL. (i) $(6,6),(4,-2)$

If $R(x,y)$ be desired midpoint, then,

$$x=\frac{6+4}{2}=\frac{10}{2}=5$$
$$y=\frac{6-2}{2}=\frac{4}{2}=2$$
$$R(x,y)=R(5,2)$$

ii) $(-5,-7),(-7,-5)$

If $R(x,y)$ be desired midpoint, then,

$$x=\frac{-5-7}{2}=\frac{-12}{2}=-6$$
$$y=\frac{-5-7}{2}=\frac{-12}{2}=-6$$
$$\therefore \quad R(x,y)=R(-6,-6)$$

iii) $(8,0),(0,-12)$

If $R(x,y)$ be desired midpoint, then,

$$x=\frac{8+0}{2}=\frac{8}{2}=4$$
$$y=\frac{-12+0}{2}=\frac{-12}{2}=-6$$
$$\therefore \quad R(x,y)=R(4,-6)$$

Objective

1. Distance between points (0, 0) and (1, 1) is:
   (a) 0 (b) 1
   (c) $\sqrt{2}$ (d) 2
2. Distance between the points (1, 0) and (0, 1) is:
   (a) 0 (b) 1
   (c) $\sqrt{2}$ (d) 2
3. Mid-point of the points (2, 2) and (0,0) is:
   (a) (1, 1) (b) (1, 0)
   (c) (0, 1) (d) (−1, −1)
4. Mid-point of the points (2, −2) and (−2, 2) is:
   (a) (2, 2) (b) (−2, −2)
   (c) (0, 0) (d) (1, 1)
5. A triangle having all sides equal is called
   (a) Isosceles (b) Scalene
   (c) Equilateral (d) None of these
6. A triangle having all sides different is called:
   (a) Isosceles (b) Scalene
   (c) Equilateral (d) None of these
7. The points P, Q and R are collinear if:
   (a) $|PQ|+|QR|=|PR|$
   (b) $|PQ|-|QR|=|PR|$
   (c) $|PQ|+|QR|=0$
   (d) None

8. The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the coordinate plane is:

(a) $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$, $d>0$

(b) $d=\sqrt{(x_1-x_2)^2-(y_1-y_2)^2}$

(c) $d=\sqrt{(x_2-x_1)^2-(y_2-y_1)^2}$

(d) $d=\sqrt{(x_1+x_2)^2-(y_1+y_2)^2}$

9. A triangle having two sides equal is called

(a) Isosceles (b) Scalene

(c) Equilateral (d) None

10. A right triangle is that in which one of the angles has measure equal to:

(a) 80° (b) 90°

(c) 45° (d) 60°

11. In a right angle triangle ABC, Pythagoras's theorem,

(a) $|AB|^2=|BC|^2+|CA|^2$ where $\angle ACB=90^\circ$.

(b) $|AB|^2=|BC|^2-|CA|^2$

(c) $|AB|^2+|BC|^2>|CA|^2$

(d) $|AB|^2-|BC|^2>|CA|^2$

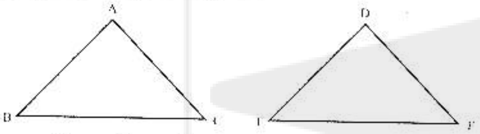
Answer key

1.	c	**2.**	c	**3.**	a	**4.**	c	**5.**	c	**6.**	b
7.	a	**8.**	a	**9.**	a	**10.**	b	**11**	a		

Unit 10

CONGRUENT TRIANGLES

Congruent Triangle



Let there be two triangles ABC and DEF. Out of the total six (1 – 1) correspondences that can be established between ΔABC and ΔDEF. One of the choices is explained below.

In the correspondence $\Delta ABC \leftrightarrow \Delta DEF$ it means.

$\angle A \leftrightarrow \angle D$ ($\angle A$ corresponds to $\angle D$)

$\angle B \leftrightarrow \angle E$ ($\angle B$ corresponds to $\angle E$)

$\angle C \leftrightarrow \angle F$ ($\angle C$ corresponds to $\angle F$)

$\overline{AB} \leftrightarrow \overline{DE}$ ($\overline{AB}$ corresponds to $\overline{DE}$)

$\overline{BC} \leftrightarrow \overline{EF}$ ($\overline{BC}$ corresponds to $\overline{EF}$)

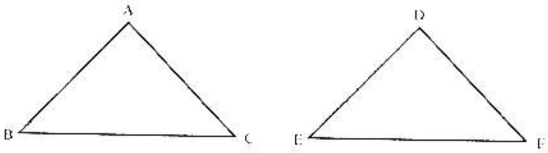
$\overline{CA} \leftrightarrow \overline{FD}$ ($\overline{CA}$ corresponds to $\overline{FD}$)

Congruency of Triangles

Two triangles are said to be congruent written symbolically as, $\cong$, if there exists a correspondence between them such that all the corresponding sides and angles are congruent i.e.

$$\text{If}\begin{cases}\overline{AB} \cong \overline{DE}\\ \overline{BC} \cong \overline{EF}\\ \overline{CA} \cong \overline{FD}\end{cases} \quad \text{and} \quad \begin{cases}\angle A \cong \angle D\\ \angle B \cong \angle E\\ \angle C \cong \angle F\end{cases}$$

Then $\Delta ABC \cong \Delta DEF$



Note

(i) These triangles are congruent w.r.t. the above mentioned choice of the (1 – 1) correspondence.

(ii) $\Delta ABC \cong \Delta ABC$

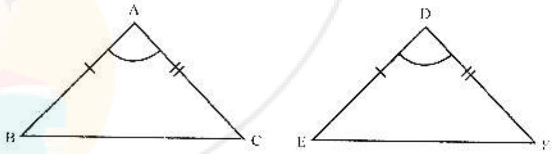
(iii) $\Delta ABC \cong \Delta DEF \Leftrightarrow \Delta DEF \cong \Delta ABC$

(iv) If $\Delta ABC \cong \Delta DEF$ and $\Delta ABC \cong \Delta PQR$, then $\Delta DEF \cong \Delta PQR$

In any correspondence of two triangles, if two sides and their included angle of one triangle are congruent to the corresponding two sides and their included angle of the other, then the triangles are congruent.

In $\Delta ABC \leftrightarrow \Delta DEF$, shown in the following figure.

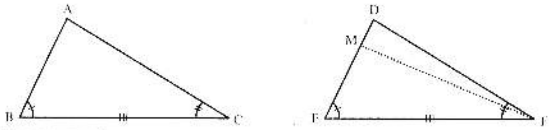
$$\text{If}\begin{cases}\overline{AB} \cong \overline{DE}\\ \angle A \cong \angle D\\ \overline{AC} \cong \overline{DF}\end{cases}$$



Then $\Delta ABC \cong ADEF$ (S.A.S. Postulate)

Theorem

In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the corresponding, side and angles of the other, then the triangles are congruent. (A.S.A $\cong$ A.S.A)



Given

In $\Delta ABC \leftrightarrow \Delta DEF$

$\angle B \cong \angle E$

$\overline{BC} \cong \overline{EF}$

$\angle C \cong \angle F$

To prove

$\Delta ABC \leftrightarrow \Delta DEF$

Construction

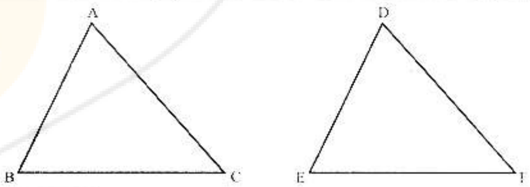
Suppose $\overline{AB} \not\cong \overline{DE}$, take a point M on $\overline{DE}$ such that $\overline{AB} \cong \overline{ME}$. Join M to F

Proof

Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta MEF$	
$\overline{AB} \cong \overline{ME}$ ....(i)	Construction
$\overline{BC} \cong \overline{EF}$ ....(ii)	Given
$\angle B \cong \angle E$...........(iii)	Given
$\therefore$ $\Delta ABC \cong \Delta MEF$	S.A.S. postulate
So, $\angle C \cong \angle MFE$	(Corresponding angles of congruent triangles)
But $\angle C \cong \angle DFE$	Given
$\therefore$ $\angle DFE \cong \angle MFE$	Both congruent to $\angle C$
This is possible only if D and M are the same points, and $\overline{ME} \cong \overline{DE}$	
So, $\overline{AB} \cong \overline{DE}$ ....(iv)	$\overline{AB} \cong \overline{ME}$ (construction) and
Thus from (i), (iii) and (iv), we have	$\overline{ME} \cong \overline{DE}$ (proved)
$\Delta ABC \cong \Delta DEF$	S.A.S. postulate

Example

In any correspondence of two triangles, if one side and any two angles of one triangle are congruent to the correspondence side and angles of the other, then the triangles are congruent. (S.A.A $\cong$ S.A.A.)



Given

In $\Delta ABC \leftrightarrow \Delta DEF$

$\overline{BC} \cong \overline{EF}$, $\angle A \cong \angle D$, $\angle B \cong \angle E$

To Prove

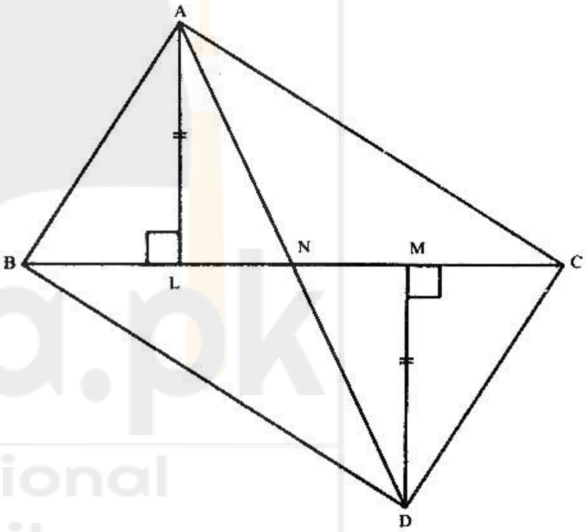
$\Delta ABC \cong \Delta DEF$

Proof

Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta DEF$	
$\angle B \cong \angle E$	Given
$\overline{BC} \cong \overline{EF}$	Given
$\angle C \cong \angle F$	$\angle A \cong \angle D$, $\angle B \cong \angle E$, (Given)
$\therefore$ $\Delta ABC \cong \Delta DEF$	A.S.A. $\cong$ A.S.A

Example

If ΔABC and ΔDCB are on the opposite sides of common base $\overline{BC}$ such that $\overline{AL} \perp \overline{BC}$, $\overline{DM} \perp \overline{BC}$, $\overline{AL} \cong \overline{DM}$, then $\overline{BC}$ bisects $\overline{AD}$.



Given

ΔABC and ΔDCB are on the opposite sides of $\overline{BC}$ such that $\overline{AL} \perp \overline{BC}$, $\overline{DM} \perp \overline{BC}$, $\overline{AL} \cong \overline{DM}$ and $\overline{AD}$ is cut by $\overline{BC}$ at N.

To Prove

$\overline{AN} \cong \overline{DN}$

Statements	Reasons
In $\Delta ALN \leftrightarrow \Delta DMN$	
$\overline{AL} \cong \overline{DM}$	Given
$\angle ALN \cong \angle DMN$	Each angle is right angle
$\angle ANL \cong \angle DNM$	Vertical angles
$\Delta ALN \cong \Delta DMN$	S.A.A. $\cong$ S.A.A
Hence $\overline{AN} \cong \overline{DN}$	Corresponding sides of $\cong$ Δs.

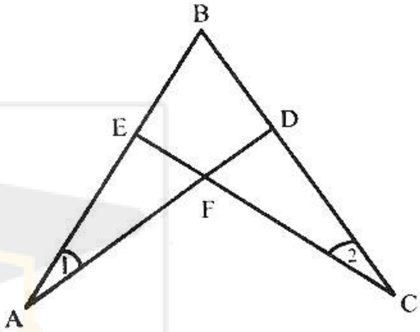
Exercise 10.1

1. In the given figure.

$\overline{AB} \cong \overline{CB}, \angle 1 \cong \angle 2.$

Prove that

$\Delta ABD \cong \Delta CBE$



Given

$\overline{AB} \cong \overline{CB}$

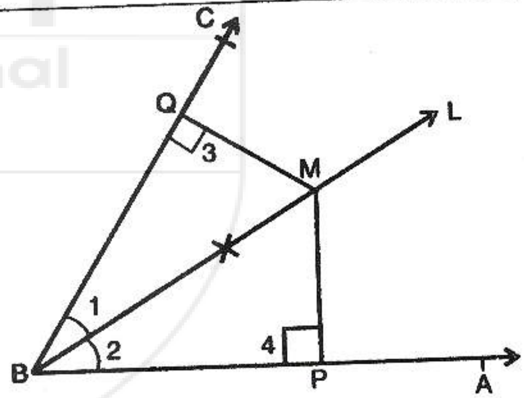
$\angle 1 = \angle 2$

To Prove

$\Delta ABD \cong \Delta CBE$

Statements	Reasons
In $\Delta ABD \leftrightarrow \Delta CBE$	
$\overline{AB} \cong \overline{CB}$	Given
$\angle 1 \cong \angle 2$	Given
$\angle ABD \cong \angle CBE$	Common angle
$\therefore$ $\Delta ABD \cong \Delta CBE$	A.S.A $\cong$ A.S.A

(2) From a point on the bisector of an angle, perpendiculars are drawn to the arms of the angle. Prove that these perpendiculars are equal in measure.



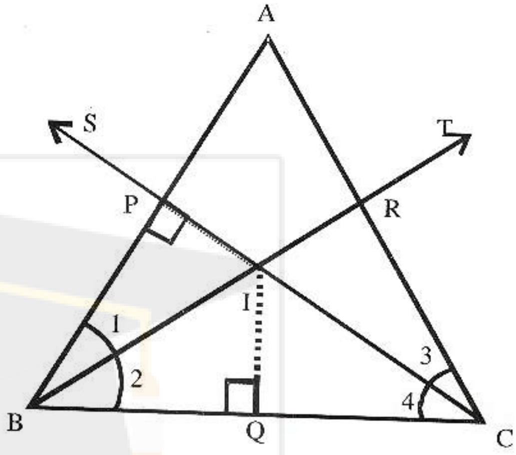
Given

$\angle ABC$, $\overrightarrow{BL}$ he bisector of $\angle ABC$, M any point on $\overrightarrow{BL}$, $\overline{MP}$ perpendicular on $\overline{AB}$, $\overline{MQ} \perp \overline{BC}$.

To Prove $\overline{MP} \cong \overline{MQ}$

Statements	Reasons
In $\Delta BMP \leftrightarrow \Delta BMQ$	
$\angle 1 \cong \angle 2$	$\overrightarrow{BL}$ bisects $\angle PBQ$
$\angle 3 \cong \angle 4$	Each = 90^o
$\overline{BM} \cong \overline{BM}$	Common
$\Delta BMP \cong \Delta BMQ$	A.S.A$\cong$ A.S.A
$\overline{PM} \cong \overline{QM}$	Corresponding sides of the congruent triangles.

(3) In a triangle ABC, the bisectors of $\angle B$ and $\angle C$ meet in a point I. Prove that I is equidistant from the three sides of ΔABC.



Given

In ΔABC, $\overrightarrow{BT}, \overrightarrow{CS}$ are the bisectors of the angles B and C respectively.

To Prove

I is equidistant from the three sides of ΔABC i.e. $\overline{IP} \cong \overline{IQ} \cong \overline{IR}$

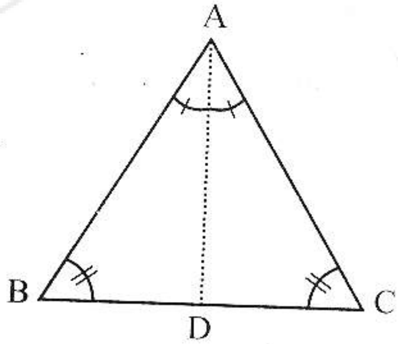
Construction

$\overline{IR} \perp \overline{AC}, \overline{IQ} \perp \overline{BC}, \overline{IP} \perp \overline{AB}$

Statements	Reasons
In $\Delta IPB \leftrightarrow \Delta IQB$	
$\angle 1 \cong \angle 2$	Given
$\angle P \cong \angle Q$	Each = 90°
$\overline{IB} \cong \overline{IB}$	Common
$\Delta IPB \cong \Delta IQB$	A.S.A ≅ A.S.A
$\overline{IP} \cong \overline{IQ}$ ....(i)	Corresponding sides of congruent triangles
Similarly $\Delta IRC \cong \Delta IQC$	
$\overline{IQ} \cong \overline{IR}$ ...(ii)	Corresponding sides of congruent triangles
$\overline{IP} \cong \overline{IQ} \cong \overline{IR}$	By (i) and (ii)

Theorem

If two angles of a triangle are congruent, then the sides opposite to them are also congruent.



Given

In ΔABC, $\angle B \cong \angle C$

To Prove

$\overline{AB} \cong \overline{AC}$

Construction

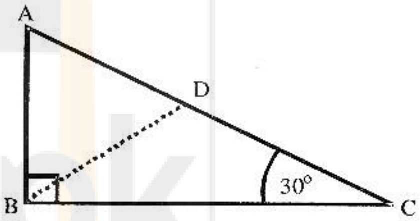
Draw the bisector of $\angle A$, meeting $\overline{BC}$ at the point D.

Proof

Statements	Reasons
In $\Delta ABD \leftrightarrow \Delta ACD$	
$\overline{AD} \cong \overline{AD}$	Common
$\angle B \cong \angle C$	Given
$\angle BAD \cong \angle CAD$	Construction
$\therefore$ $\Delta ABD \cong \Delta ACD$	S.A.A. $\cong$ S.A.A.
Hence $\overline{AB} \cong \overline{AC}$	Corresponding sides of congruent triangles

Example

If one angle of a right triangle is of 30°, the hypotenuse is twice as long as the side opposite to the angle.



Given

In ΔABC, $m\angle B = 90^\circ$ and $m\angle C = 30^\circ$

To Prove

$m\overline{AC} = 2m\overline{AB}$

Construction

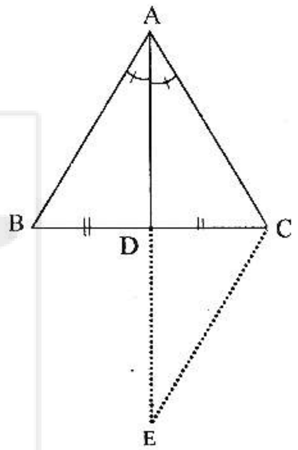
At B, construct $\angle CBD$ of 30°. Let $\overline{BD}$ cut $\overline{AC}$ at the point D.

Proof

Statements	Reasons
In ΔABD, $m\angle A = 60^\circ$	$m\angle ABC = 90^\circ$, $m\angle C = 30^\circ$
$m\angle ABD = m\angle ABC - m\angle CBD = 60^\circ$	$m\angle ABC = 90^\circ$, $m\angle CBD = 30^\circ$
$\therefore$ $m\angle ADB = 60^\circ$	Sum of measures of $\angle$s of aΔ is 180°
$\therefore$ ΔABD is equilateral	Each of its angles is equal to 60°
$\therefore$ $\overline{AB} \cong \overline{BD} \cong \overline{AD}$	Sides of equilateral Δ
In ΔBCD, $\overline{BD} \cong \overline{CD}$	$\angle C = \angle CBD$ (each of 30°).
Thus $\left.\begin{aligned} m\overline{AC} &= m\overline{AD} + m\overline{CD} \\ &= m\overline{AB} + m\overline{AB} \\ &= 2(m\overline{AB}) \end{aligned}\right\}$	$\overline{AD} \cong \overline{AB}$ and $\overline{CD} \cong \overline{BD} \cong \overline{AB}$

Example

If the bisector of an angle of a triangle bisects the side opposite to it, the triangle is isosceles.



Given

In ΔABC, $\overline{AD}$ bisects $\angle A$ and $\overline{BD} \cong \overline{CD}$

To Prove

$\overline{AB} \cong \overline{AC}$

Construction

Produce $\overline{AD}$ to E, and take $\overline{ED} \cong \overline{AD}$.

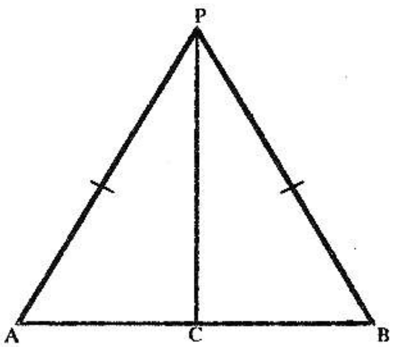
joint C to E

Proof

Statements		Reasons
In	$\Delta ABD \leftrightarrow \Delta EDC$	
	$\overline{AD} \cong \overline{ED}$	Construction
	$\angle ADB \cong \angle EDC$	Vertical angles
	$\overline{BD} \cong \overline{CD}$	Given
$\therefore$	$\Delta ADB \cong \Delta EDC$	S.A.S. Postulate
$\therefore$	$\overline{AB} \cong \overline{EC}$ ....(1)	Corresponding sides of $\cong \Delta$s
and	$\angle BAD \cong \angle E$	Corresponding angles of $\cong \Delta$s
But	$\angle BAD \cong \angle CAD$	Given
$\therefore$	$\angle E \cong \angle CAD$	Each $\cong \angle BAD$
In	ΔACE, $\overline{AC} \cong \overline{EC}$ ....(2)	$\angle E \cong \angle CAD$ (proved)
Hence	$\overline{AB} \cong \overline{AC}$	From (1) and (2)

Exercise 10.2

Prove that a point, which is equidistant from the end points of a line segment, is on the right bisector of the line segment.



Given

$\overline{AB}$ is a line segment. Point P is such that $\overline{PA} \cong \overline{PB}$

To Prove

Point P is on the right bisector of $\overline{AB}$.

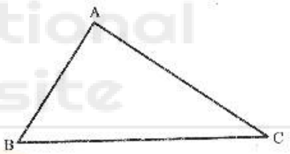
Construction

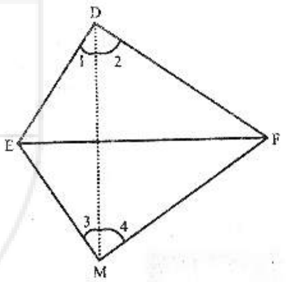
Join P to C, the midpoint of $\overline{AB}$

Proof

Statements	Reasons
In $\Delta ACP \leftrightarrow \Delta BCP$	
$\overline{PA} \cong \overline{PB}$	Given
$\overline{PC} \cong \overline{PC}$	Common
$\overline{AC} \cong \overline{BC}$	Construction
$\Delta ACP \cong \Delta BCP$	S.S.S $\cong$ S.S.S
$\angle ACP \cong \angle BCP$ ...(i)	Corresponding angles of congruent triangles
But $m\angle ACP + m\angle BCP = 180^o$...(ii)	supplementary angles,
$m\angle ACP = m\angle BCP = 90^o$	From (i) and (ii)
or $\overline{PC} \perp \overline{AB}$ ....(iii)	$m\angle ACP = 90^o$ (proved)
Also $\overline{CA} \cong \overline{CB}$ ....(iv)	construction
$\therefore$ $\overline{PC}$ is a right bisector Of $\overline{AB}$ i.e, the point P is on the right bisector of $\overline{AB}$.	from (iii) and (vi)

Theorem





In a correspondence of two triangles, if three sides of one triangle are congruent to the corresponding three sides of the other, then the two triangles are congruent.

(S.S.S. $\cong$ S.S.S.)

Given

In $\Delta ABC \leftrightarrow \Delta DEF$

$\overline{AB} \cong \overline{DE}, \overline{BC} \cong \overline{EF}$ and $\overline{CA} \cong \overline{FD}$

To Prove

$\Delta ABC \cong \Delta DEF$

Construction

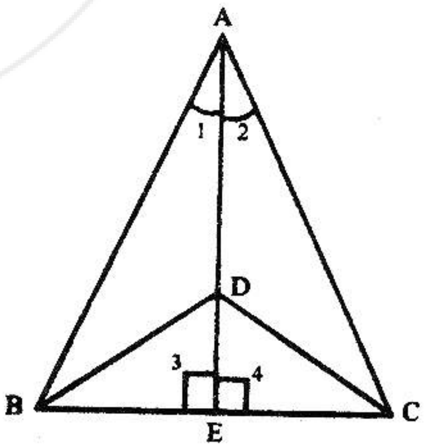
Suppose that in ΔDEF the side $\overline{EF}$ is not smaller than any of the remaining two sides. On $\overline{EF}$ construct a ΔMEF in which, $\angle FEM \cong \angle B$ and $\overline{ME} \cong \overline{AB}$. Join D and M. As shown in the above figures we label some of the angles as 1,2,3 and 4.

Proof

Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta MEF$	
$\overline{BC} \cong \overline{EF}$	Given
$\angle B \cong \angle FEM$	Construction
$\overline{AB} \cong \overline{ME}$	Construction
$\therefore$ $\Delta ABC \cong \Delta MEF$	S.A.S postulate
and $\overline{CA} \cong \overline{FM}$ ……………….(i)	(Corresponding sides of congruent triangles)
Also $\overline{CA} \cong \overline{FD}$ ……………….(ii)	Given
$\therefore$ $\overline{FM} \cong \overline{FD}$	From (i) and (ii)
In ΔFDM	
$\angle 2 \cong \angle 4$……………….(iii)	$\overline{FM} \cong \overline{FD}$ (proved)
Similarly $\angle 1 \cong \angle 3$ …………….(iv)	
$\therefore$ $m\angle 2 + m\angle 1 = m\angle 4 + m\angle 3$	{from (iii) and (iv)}
$\therefore$ $m\angle EDF = m\angle EMF$	
Now, In $\Delta DEF \leftrightarrow \Delta MEF$	
$\overline{FD} \cong \overline{FM}$	Proved
And $m\angle EDF \cong m\angle EMF$	Proved
$\overline{DE} \cong \overline{ME}$	Each one $\cong \overline{AB}$
$\therefore$ $\Delta DEF \cong \Delta MEF$	S.A.S postulate
Also $\Delta ABC \cong \Delta MEF$	Proved
Hence $\Delta ABC \cong \Delta DEF$	Each $\Delta \cong \Delta MEF$ (Proved)

Example

If two isosceles triangles are formed on the same side of their common base, the line through their vertices would be the right bisector of their common base.



Given

ΔABC and ΔDBC are formed on the same side of $\overline{BC}$ such that

$\overline{AB} \cong \overline{AC}$, $\overline{DB} \cong \overline{DC}$, $\overline{AD}$ meets $\overline{BC}$ at E.

To prove

$\overline{BE} \cong \overline{CE}$, $\overline{AE} \perp \overline{BC}$

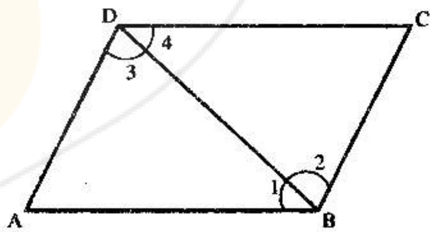
Proof

Statements	Reasons
In $\Delta ADB \leftrightarrow \Delta ADC$	
$\overline{AB} \cong \overline{AC}$	Given
$\overline{DB} \cong \overline{DC}$	Given
$\overline{AD} \cong \overline{AD}$	Common
$\therefore$ $\Delta ADB \cong \Delta ADC$	S.S.S $\cong$ S.S.S.
$\therefore$ $\angle 1 \cong \angle 2$	Corresponding angles of $\cong \Delta s$
In $\Delta ABE \leftrightarrow \Delta ACE$	
$\overline{AB} \cong \overline{AC}$	Given
$\angle 1 \cong \angle 2$	Proved
$\overline{AE} \cong \overline{AE}$	Common
$\therefore$ $\Delta ABE \cong \Delta ACE$	S.A.S. postulate
$\therefore$ $\overline{BE} \cong \overline{CE}$	Corresponding sides of $\cong \Delta s$
$\angle 3 \cong \angle 4$ I	Corresponding angles of $\cong \Delta s$
$m\angle 3 + m\angle 4 = 180^\circ$.........II	Supplementary angles Postulate
$\therefore$ $m\angle 3 = m\angle 4 = 90^\circ$	From I and II
Hence $\overline{AE} \perp \overline{BC}$	

Corollary: An equilateral triangle is an equiangular triangle.

Exercise 10.3

Q1. In the figure, $\overline{AB} \cong \overline{DC}$, $\overline{AD} \cong \overline{BC}$.
Prove that $\angle A \cong \angle C$, $\angle ABC \cong \angle ADC$.



Given $\overline{AB} \cong \overline{DC}$

$\overline{AD} \cong \overline{BC}$

To prove $\angle A \cong \angle C$

$\angle ABC \cong \angle ADC$

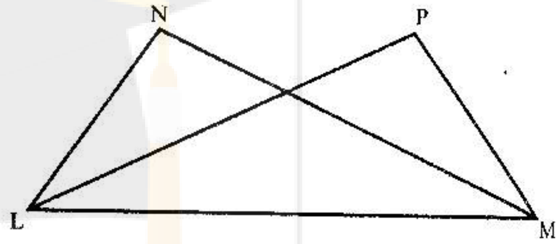
Proof

Statements	Reasons
In $\Delta ABD \leftrightarrow \Delta CBD$	
$\overline{AB} \cong \overline{DC}$	Given
$\overline{AD} \cong \overline{BC}$	Given

	$\overline{BD} \cong \overline{BD}$	Common
$\therefore$	$\Delta ABD \cong \Delta CBD$	S.S.S $\cong$ S.S.S
	$\angle A \cong \angle C$	
	$\angle 1 \cong \angle 4$ ....(i)	Corresponding angles of congruent triangles
	$\angle 2 \cong \angle 3$ ....(ii)	Corresponding angles of congruent triangles
	$\angle 1 + \angle 2 = \angle 3 + \angle 4$	Adding (i) and (ii)
	$\angle ABC \cong \angle ADC$	

2. In the figure, $\overline{LN} \cong \overline{MP}$, $\overline{MN} \cong \overline{LP}$.

Prove that $\angle N \cong \angle P, \angle NML \cong \angle PLM$.



Given

$\overline{LN} \cong \overline{MP}$

$\overline{LP} \cong \overline{MN}$

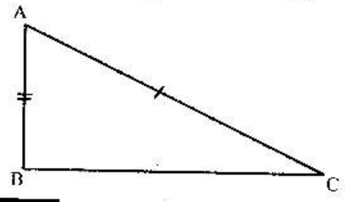
To prove

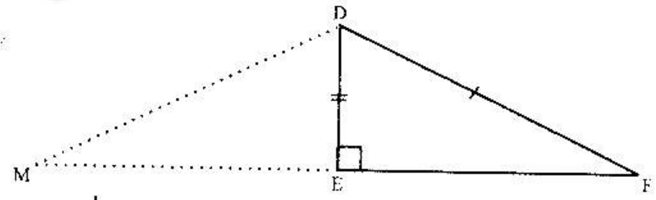
$\angle N \cong \angle P$, $\angle NML \cong \angle PLM$

	Statements	Reasons
In	$\Delta LMN \leftrightarrow \Delta LMP$	
	$LM \cong \overline{MP}$	Given
	$\overline{LP} \cong \overline{MN}$	Given
	$\overline{LM} \cong LM$	Common
	$\Delta LMN \cong \Delta LPM$	S.S.S $\cong$ S.S.S
	$\angle N = \angle P$	Corresponding angles of congruent triangles
	$\angle NML \cong \angle PLM$	Corresponding angles of congruent triangles

Theorem

If in the correspondence of the two right-angled triangles, the hypotenuse and one side of one triangle are congruent to the hypotenuse and the corresponding side of the other, then the triangles are congruent. (H.S $\cong$ H.S)





Given

In $\Delta ABC \leftrightarrow \Delta DEF$	$\angle B \cong \angle E$ (right angles)
	$\overline{CA} \cong \overline{FD}$, $\overline{AB} \cong \overline{DE}$

To Prove $\Delta ABC \cong \Delta DEF$

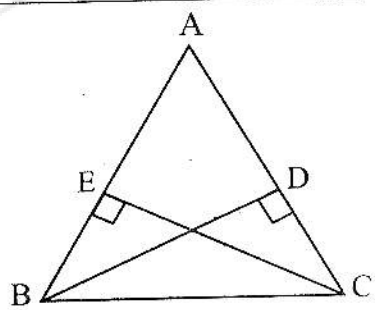
Construction Produce $\overline{FE}$ to a point M such that $\overline{EM} \cong \overline{BC}$ and join the points D and M.

Proof

Statements	Reasons
In $m\angle DEF + m\angle DEM = 180^\circ \ldots(i)$	(Supplementary angles)
Now $m\angle DEF = 90^\circ \ldots\ldots\ldots\ldots\ldots\ldots(ii)$	(Given)
$\therefore$ $m\angle DEM = 90^\circ$	{from (i) and (ii)}
In $\Delta ABC \leftrightarrow \Delta DEM$	
$\overline{BC} \cong \overline{EM}$	(construction)
$\angle ABC \cong \angle DEM$	(each $\angle$ equal to 90°)
$\overline{AB} \cong \overline{DE}$	(given)
$\therefore$ $\Delta ABC \cong \Delta DEM$	S.A.S. postulate
And $\angle C \cong \angle M$	(Corresponding angles of congruent triangles)
$\overline{CA} \cong \overline{MD}$	(Corresponding sides of congruent triangles)
But $\overline{CA} \cong \overline{FD}$	(given)
$\therefore$ $\overline{MD} \cong \overline{FD}$	Each is congruent to $\overline{CA}$
In ΔDMF	
$\angle F \cong \angle M$	$\overline{FD} \cong \overline{MD}$ (Proved)
But $\angle C \cong \angle M$	(proved)
$\angle C \cong \angle F$	(each is congruent to $\angle M$)
In $\Delta ABC \leftrightarrow \Delta DEF$	
$\overline{AB} \cong \overline{DE}$	(given)
$\angle ABC \cong \angle DEF$	(given)
$\angle C \cong \angle F$	(proved)
Hence $\Delta ABC \cong \Delta DEF$	(S.A.A $\cong$ S.A.A)

Example

If perpendiculars from two vertices of a triangle to the opposite sides are congruent, then the triangle is isosceles.



Given

In ΔABC, $\overline{BD} \perp \overline{AC}$, $\overline{CE} \perp \overline{AB}$

Such that $\overline{BD} \cong \overline{CE}$

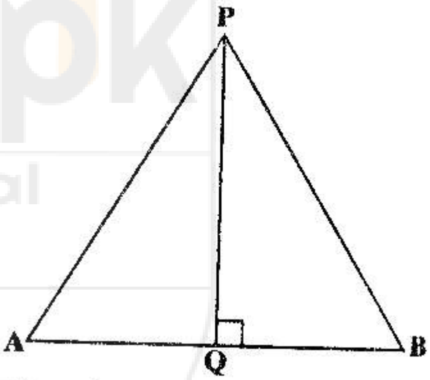
To Prove $\overline{AB} \cong \overline{AC}$

Proof

Statements	Reasons
In $\Delta BCD \leftrightarrow \Delta CBE$	
$\angle BDC \cong \angle BEC$	$\overline{BD} \perp \overline{AC}, \overline{CE} \perp \overline{AB}$ (given) $\Rightarrow$ each angle = 90^o
$\overline{BC} \cong \overline{BC}$	Common hypotenuse
$\overline{BD} \cong \overline{CE}$	Given
$\Delta BCD \cong \Delta CBE$	H.S. ≅ H.S.
$\angle BCD \cong \angle CBE$	Corresponding angles of ≅ Δs.
Thus $\angle BCA \cong \angle CBA$	
Hence $\overline{AB} \cong \overline{AC}$	In ΔABC, $\angle BCA \cong \angle CBA$

Exercise 10.4

1. In ΔPAB of figure, $\overline{PQ} \perp \overline{AB}$ and $\overline{PA} \cong \overline{PB}$, prove that $\overline{AQ} \cong \overline{BQ}$ and $\angle APQ \cong \angle BPQ$.



Given

In ΔPAB, $\overline{PQ} \perp \overline{AB}$ and $\overline{PA} \cong \overline{PB}$

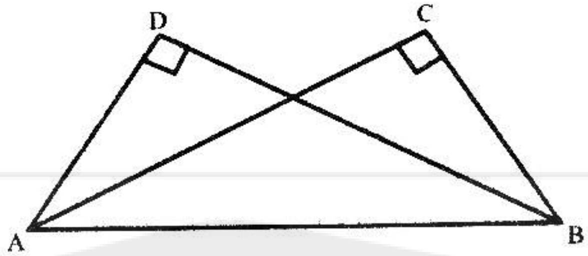
To Prove

$\overline{AQ} \cong \overline{BQ}$ and $\angle APQ \cong \angle BPQ$

Proof

Statements	Reasons
In $\Delta APQ \leftrightarrow \Delta BPQ$	
$\overline{PA} \cong \overline{PB}$	Given
$\overline{PQ} \cong \overline{PQ}$	Common
$\therefore$ $\Delta PAQ \cong \Delta PBQ$	H.S ≅ H.S
$\therefore$ $\overline{AQ} \cong \overline{BQ}$	Corresponding sides of congruent triangles
$\angle APQ \cong \angle BPQ$	Corresponding angles of the congruent triangles.

2. In the figure, $m\angle C = m\angle D = 90^o$ and $\overline{BC} \cong \overline{AD}$. Prove that $\overline{AC} \cong \overline{BD}$ and $\angle BAC \cong \angle ABD$.



Given

$m\angle C = m\angle D = 90^\circ$

$\overline{BC} \cong \overline{AD}$

To Prove

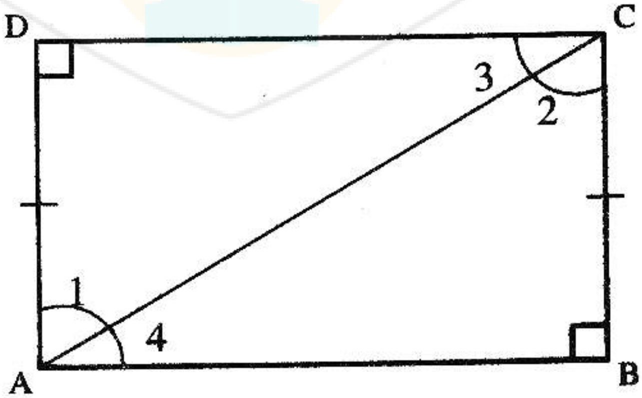
$\overline{AC} \cong \overline{BD}$

$\angle BAC \cong \angle ABD$

Proof

Statements	Reasons
In $\Delta ABC \leftrightarrow \Delta ABD$	
$m\angle C \cong m\angle D$	Each of 90°
$\overline{BC} \cong \overline{AD}$	Given
$\overline{AB} \cong \overline{AB}$	Common
$\therefore$ $\Delta ABC \cong \Delta ABD$	H.S $\cong$ H.S
$\overline{AC} \cong \overline{BD}$	Corresponding sides of congruent triangles
$\therefore$ $\angle BAC \cong \angle ABD$	Corresponding angles of the congruent triangles

3. In the figure, $m\angle B = m\angle D = 90^\circ$ and $\overline{AD} \cong \overline{BC}$. Prove that ABCD is a rectangle.



Given

$m\angle B = m\angle D = 90^\circ$, $\overline{AD} \cong \overline{BC}$

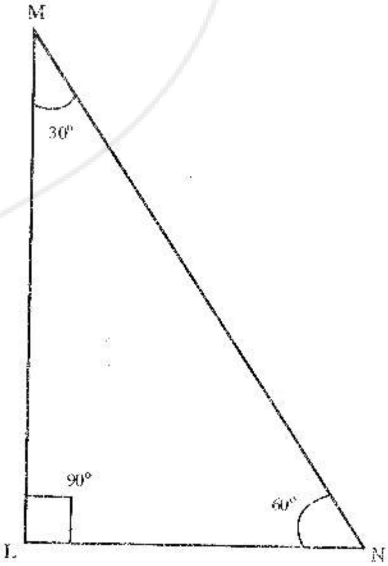
Proof

ABCD is a rectangle

Statements		Reasons
In	$\Delta ABC \leftrightarrow \Delta ADC$	
	$m\angle B \cong m\angle D$	Each of 90°
	$\overline{AD} \cong \overline{BC}$	Given
	$\overline{AC} \cong \overline{AC}$	Common
$\therefore$	$\Delta ABC \cong \Delta ADC$	H.S $\cong$ H.S
	$\overline{AB} \cong \overline{DC}$	
	$\angle 1 \cong \angle 2$ ...(i)	
	$\angle 4 \cong \angle 3$ ...(ii)	
	$\angle 1 + \angle 4 = \angle 2 + m\angle 3$	
	$\angle A = \angle C = 90^\circ$	
	ABCD is a rectangle	By (i) and (ii)

4. Which of the following are true and which are false?

(i) A ray has two end points.
(ii) In a triangle, there can be only one right angle.
(iii) Three points are said to be collinear if they lie on same line.
(iv) Two parallel lines intersect at a point.
(v) Two lines can intersect only in one point.
(vi) A triangle of congruent sides has non-congruent angles.



Answers

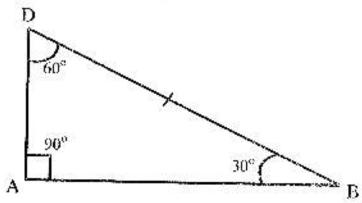
(i)	False	(ii)	True	(iii)	True
(iv)	False	(v)	True	(vi)	False

5. If $\Delta ABC \cong \Delta LMN$, then

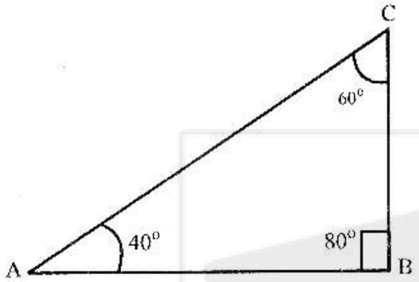
(i) $m\angle M \cong$ ....
(ii) $m\angle N \cong$ ....
(iii) $m\angle A \cong$ ....

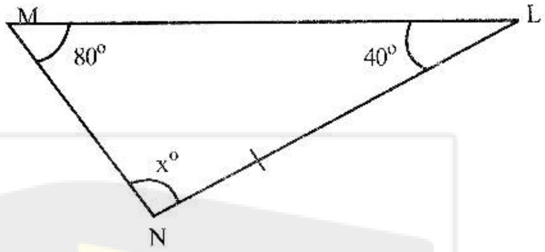
Answers

(i) $m\angle M \cong m\angle B$
(ii) $m\angle N \cong m\angle C$
(iii) $m\angle A \cong m\angle L$



6. If $\Delta ABC \cong \Delta LMN$, then find the unknown x.

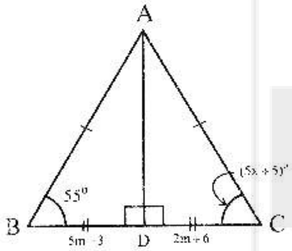




Answers

$x = 60°$

7. Find the value of unknowns for the given congruent triangles.



$\Delta ABD \cong \Delta ACD$

$\overline{BD} \cong \overline{DC}$

$\Rightarrow \quad 5m - 3 = 2m + 6$

$5m - 2m = 3 + 6$

$3m = 9$

$m = \dfrac{9}{3} = 3$

Also

$\angle ACD \cong \angle ABD \Rightarrow$

Angles opposite to congruent sides are congruent

$5x + 5 = 55$

$5x = 55 - 5$

$5x = 50$

$x = \dfrac{50}{5}$

$x = 10$

8. If $\Delta PQR \cong \Delta ABC$

, then find the unknowns.

$\Delta PQR \cong \Delta ABC$

$\overline{PQ} \cong \overline{AB}$

$x = 3$

$\overline{BC} \cong \overline{QR}$

$\Rightarrow z = 4$ cm

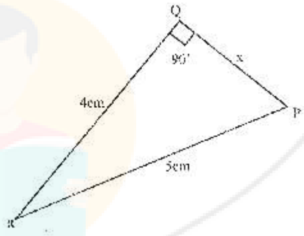
$\overline{AC} \cong \overline{PR}$

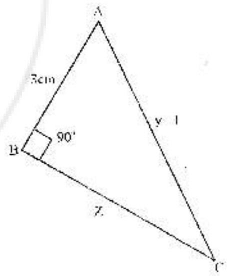
$y - 1 = 5$

$y = 5 + 1$

$y = 6$cm

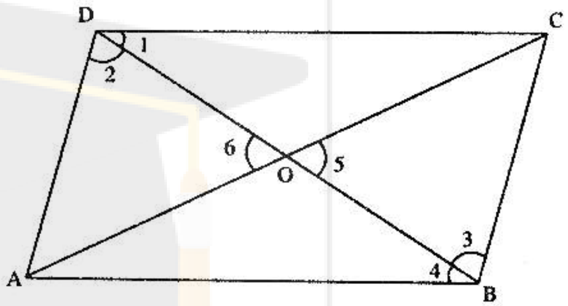
$\therefore x = 3$cm, $y = 6$cm, $z = 4$cm





Unit 11

PARALLELOGRAMS AND TRIANGLES



Theorem

In a parallelogram

(i) Opposite sides are congruent.

(ii) Opposite angles are congruent.

(iii) The diagonals bisect each other.

Given

In a quadrilateral ABCD, $\overline{AB} \parallel \overline{DC}, \overline{BC} \parallel \overline{AD}$ and the diagonals $\overline{AC}$, $\overline{BD}$ meet each other at point O.

To Prove

(i) $\overline{AB} \cong \overline{DC}$. $\overline{AD} \cong \overline{BC}$

(ii) $\angle ADC \cong \angle ABC, \angle BAD \cong \angle BCD$

(iii) $\overline{OA} \cong \overline{OC}$. $\overline{OB} \cong \overline{OD}$

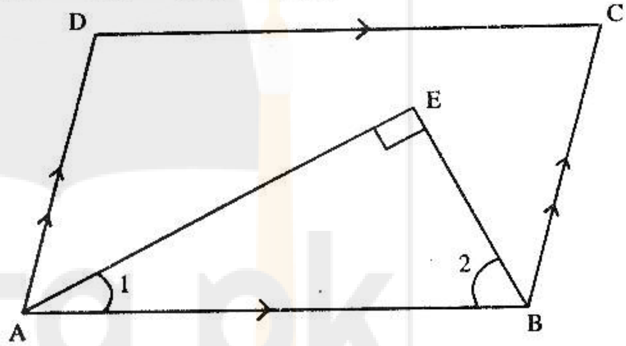
Construction

In the figure as shown, we label the angles as $\angle 1$, $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$ and $\angle 6$.

Proof

Statements	Reasons
(i) In $\Delta ABD \leftrightarrow \Delta CDB$	
$\angle 4 \cong \angle 1$	Alternate angles
$\overline{BD} \cong \overline{BD}$	Common
$\angle 2 \cong \angle 3$	Alternate angles
$\therefore$ $\Delta ABD \cong \Delta CDB$	A.S.A. $\cong$ A.S.A.
So, $\overline{AB} \cong \overline{DC}$. $\overline{AD} \cong \overline{BC}$	(corresponding sides of congruent triangles)
and $\angle A \cong \angle C$	(corresponding angles of congruent triangles)
(ii) Since	
$\angle 1 \cong \angle 4$ ......(a)	Proved
and $\angle 2 \cong \angle 3$ ......(b)	Proved
$\therefore$ $m\angle 1 + m\angle 2 = m\angle 4 + m\angle 3$	From (a) and (b)
or $m\angle ADC = m\angle ABC$	
or $\angle ADC \cong \angle ABC$	

and $\angle BAD = \angle BCD$	Proved in (i)
(iii) In $\Delta BOC \leftrightarrow \Delta DOA$	
$\overline{BC} \cong \overline{AD}$	Proved in (i)
$\angle 5 \cong \angle 6$	Vertical angles
$\angle 3 \cong \angle 2$	Proved
$\therefore$ $\Delta BOC \cong \Delta DOA$	A.A.S $\cong$ A.A.S
Hence $\overline{OC} \cong \overline{OA}$, $\overline{OB} \cong \overline{OD}$	Corresponding sides of congruent triangles)



Corollary

Each diagonal of a parallelogram bisects it into two congruent triangles.

Example

The bisectors of two angles on the same side of a parallelogram cut each other at right angles.

Given

A parallelogram ABCD, in which

$\overline{AB} \parallel \overline{DC}$, $\overline{AD} \parallel \overline{BC}$

The bisectors of $\angle A$ and $\angle B$ cut each other at E.

To prove

$m\angle E = 90^\circ$

Construction

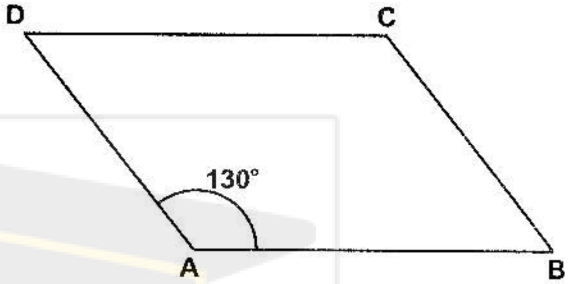
Name the angles $\angle 1$ and $\angle 2$ as shown in the figure.

Proof

Statements	Reasons
$m\angle 1 + m\angle 2$ $= \frac{1}{2}(m\angle BAD + m\angle ABC)$	$\left\{ m\angle 1 = \frac{1}{2} m\angle BAD, \; m\angle 2 = \frac{1}{2} mABC \right.$
$= \frac{1}{2}(180^\circ)$ $= 90^\circ$	Int.angles on the same side of $\overline{AB}$ Which cuts $\parallel$ segments $\overline{AD}$ and $\overline{BC}$ are supplementary.
Hence in ΔABE, $m\angle E = 90^\circ$	$m\angle 1 + m\angle 2 = 90^\circ$ (proved)

EXERCISE 11.1

(1) One angle of a parallelogram is 130°. Find the measures of its remaining angles.



Given

ABCD is a parallelogram that

$m\angle A = 130^\circ$

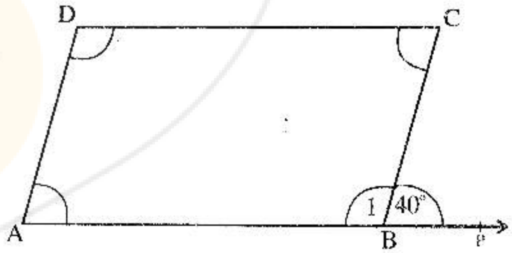
To Prove

(Required) To find the measures of $\angle B$, $\angle C$, $\angle D$

Proof

Statements	Reasons
$m\angle C = m\angle A$	Opposite angles of parallelogram.
$m\angle C = 130^\circ$	Given, $m\angle A = 130^\circ$
$m\angle B + m\angle A = 180^\circ$	$\overline{AD} \parallel \overline{BC}$ and $\overline{AB}$ is transversal. $\therefore$ sum of interior angles.
$m\angle B + 130^\circ = 180^\circ$	Given $m\angle A = 130^\circ$
$m\angle B = 180^\circ - 130^\circ$	
$m\angle B = 50^\circ$	
$m\angle D = m\angle B$	Opp. angles
$m\angle D = 50^\circ$	As $m\angle B = 50^\circ$
$\therefore$ $m\angle B = 50^\circ$, $m\angle C = 130^\circ$, $m\angle D = 50^\circ$	

(2) One exterior angle formed on producing one side of a parallelogram is 40°. Find the measures of its interior angles.



Given

ABCD is a parallelogram, side AB has been produced to p to form exterior angle $m\angle CBP = 40^\circ$ and name the interior angles as $\angle 1$, $\angle C$, $\angle D$, $\angle A$.

Required

To find the degree measures of $\angle 1$, $\angle C$, $\angle D$, $\angle A$

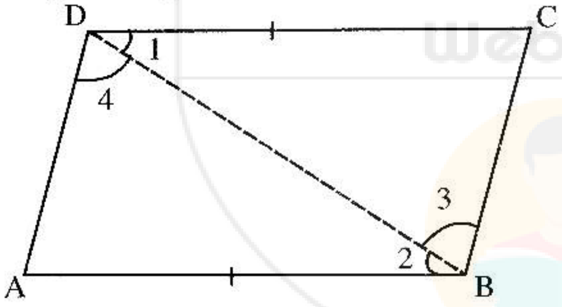
Proof

Statements	Reasons
$m\angle 1 + m\angle CBP = 180^\circ$	Supp.angles.
$m\angle 1 + 40^\circ = 180^\circ$	$m\angle CBP = 40^\circ$ given

$\therefore \quad m\angle 1 = 180^\circ - 40^\circ$	
$m\angle 1 = 140^\circ \quad$ (i)	
$m\angle D = m\angle 1$	Opp.angles of llm
$m\angle D = 140^\circ$.....(ii)	From (i)
$m\angle A + m\angle 1 = 180^\circ$	$\overline{AD} \parallel \overline{BC}$ and $\overline{AB}$ is transversal. (Interior angles)
$m\angle A + 140^\circ = 180^\circ$	From (i)
$m\angle A = 180^\circ - 140^\circ$	
$m\angle A = 40^\circ$..............(iii)	
$m\angle C = m\angle A$	Opp. angles
$m\angle C = 40^\circ$	From (iii)
Thus $m\angle 1 = 140^\circ$, $m\angle C = 40^\circ$	

Theorem

If two opposite sides of a quadrilateral are congruent and parallel, it is a parallelogram.



Given

In a quadrilateral ABCD,

$\overline{AB} \cong \overline{DC}$ and $\overline{AB} \parallel \overline{DC}$

To prove

ABCD is a parallelogram.

Construction

Join the point B to D and in the figure, name the angles as indicated:

$\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$

Proof

Statements	Reasons
In $\Delta ABD \leftrightarrow \Delta CDB$	
$\overline{AB} \cong \overline{DC}$	Given
$\angle 2 \cong \angle 1$	Alternate angles
$\overline{BD} \cong \overline{BD}$	Common
$\therefore \quad \Delta ABD \cong \Delta CDB$	S.A.S. postulate
Now $\angle 4 \cong \angle 3$ .....(i)	(corresponding angles of congruent triangles)
$\therefore \quad \overline{AD} \parallel \overline{BC}$ ... .(ii)	From (i)

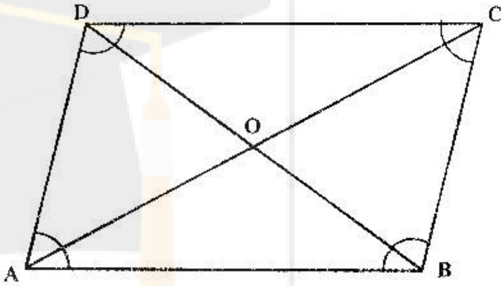
and $\overline{AD} \cong \overline{BC}$ ....(iii)	Corresponding sides of congruent Δs
Also $\overline{AB} \parallel \overline{DC}$ ....(iv)	Given
Hence ABCD is a parallelogram	From (ii) – (iv)

EXERCISE 11.2

(1) Prove that a quadrilateral is a parallelogram if its

(a) Opposite angles are congruent.

(b) Diagonals bisect each other.



Given Given ABCD is a quadrilateral.

$m\angle A = m\angle C$,

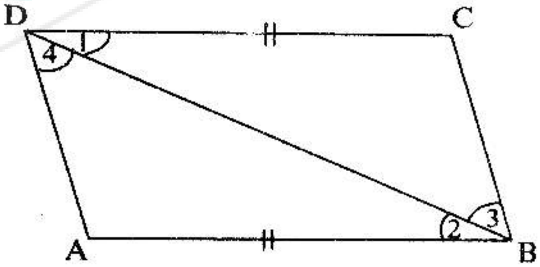
$m\angle B = m\angle D$

To prove ABCD is a parallelogram.

Proof

Statements	Reasons
$m\angle A = m\angle C$ (i)	Given
$m\angle B = m\angle D$ (ii)	Given
Now	
$m\angle A + m\angle B + m\angle C + m\angle D = 360^\circ$	Angles of a quad.
$m\angle A + m\angle B + m\angle A + m\angle B = 360^\circ$	From (i), (ii)
$m\angle A + m\angle A + m\angle B + m\angle B = 360^\circ$	Rearranging
$2m\angle A + 2m\angle B = 360^\circ$	
$(m\angle A + m\angle B) = 360^\circ/2 = 180^\circ$	Dividing by 2
$\therefore$ $\overline{AD} \parallel \overline{BC}$	As $m\angle A + m\angle B = 180^\circ$
Similarly it can be	(sum of interior angles)
Proved that $\overline{AB} \parallel \overline{CD}$	
Hence ABCD is a parallelogram.	

(2) prove that a quadrilateral is a parallelogram if its opposite sides are congruent.



Given

In quadrilateral

ABCD, $\overline{AB} \cong \overline{DC}$,

$\overline{AD} \cong \overline{BC}$

Required

ABCD is a ∥ gm

$\overline{AB} \parallel \overline{CD}$, $\overline{AD} \parallel \overline{BC}$

Construction

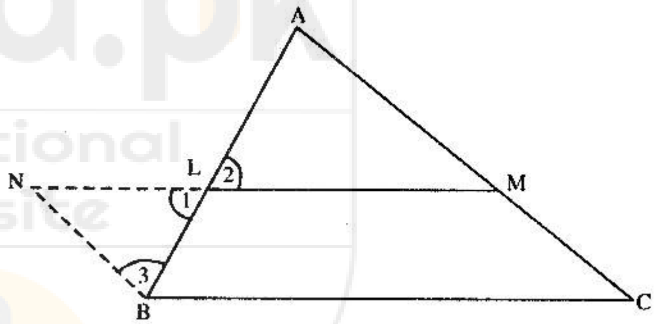
Join point B to D and name the angles $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$

Proof

Statements			Reasons
	$\Delta ABD \leftrightarrow \Delta CDB$		
	$\overline{AD} \cong \overline{CB}$		Given
	$\overline{AB} \cong \overline{CD}$		Given
	$\overline{BD} \cong \overline{BD}$		Common
$\therefore$	$\Delta ABD \cong \Delta CDB$		S.S.S $\cong$ S.S.S
So	$\angle 2 \cong \angle 1$	(i)	Corresponding angles of Congruent triangles
	$\angle 4 \cong \angle 3$	(ii)	Alternate angles
Hence	$\overline{AB} \parallel \overline{CD}$	(iii)	$\angle 2$ and $\angle 1$ are congruent
Similarly	$\overline{BC} \parallel \overline{AD}$	(iv)	Alternate angles $\angle 3$, $\angle 4$ congruent
$\therefore$	ABCD is a parallelogram.		From iii, iv

Theorem

The line segment, joining the mid-points of two sides of a triangle, is parallel to the third side and is equal to one half if its length.



Given In ΔABC, the mid-points of $\overline{AB}$ and $\overline{AC}$ are L and M respectively.

To Prove

$$\overline{LM} \parallel \overline{BC} \text{ and } m\overline{LM} = \frac{1}{2} m\overline{BC}$$

Construction

Join M to L and produce $\overline{ML}$ to N such that $\overline{ML} \cong \overline{LN}$ .Join N to B. and in the figures name the angles $\angle 1$, $\angle 2$, $\angle 3$ and $\angle$ 4 as shown.

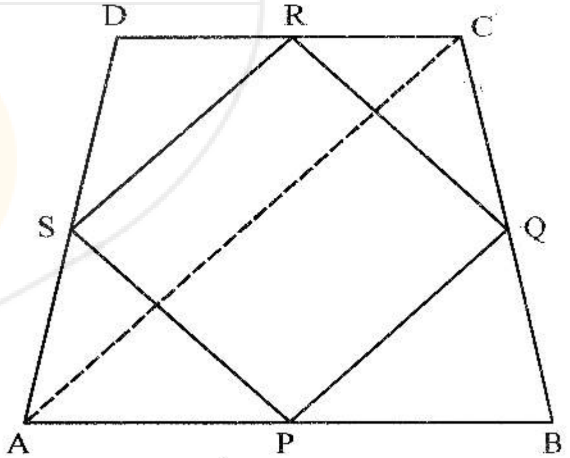
Proof

Statements		Reasons
In	$\Delta BLN \leftrightarrow \Delta ALM$	
	$\overline{BL} \cong \overline{AL}$,	Given
	$\angle 1 \cong \angle 2$	Vertical angles
	$\overline{NL} \cong \overline{ML}$	Construction

$\therefore$	$\Delta BLN \cong \Delta ALM$	S.A.S. postulate
	$\angle A \cong \angle 3$ ......(i)	(corresponding angles of congruent triangles)
and	$\overline{NB} \cong \overline{AM}$ .....(ii)	(corresponding sides of congruent triangles)
But	$\overline{NB} \parallel \overline{AM}$	From (i), alternate $\angle$s
Thus	$\overline{NB} \parallel \overline{MC}$ ....(iii)	(M is a point of $\overline{AC}$)
	$\overline{MC} \cong \overline{AM}$ ....(iv)	Given
	$\overline{NB} \cong \overline{MC}$ ... (v)	{from (ii) and (iv)}
$\therefore$	BCMN is a parallelogram	From (iii) and (v)
$\therefore$	$\overline{BC} \parallel \overline{LM}$ or $\overline{BC} \parallel \overline{NL}$	(Opposite sides of a parallelogram BCMN)
	$\overline{BC} \cong \overline{NM}$ .....(vi)	(Opposite sides of parallelogram)
	$m\overline{LM} = \frac{1}{2}\, m\overline{NM}$ .....(vii)	Construction
Hence	$m\overline{LM} = \frac{1}{2}\, m\overline{BC}$	{from (vi) and (vii)}

Example

The line segments, joining the mid-points of the sides of a quadrilateral, taken in order, form a parallelogram.



Given

A quadrilateral ABCD, in which P is the mid-point of $\overline{AB}$, Q is the mid-point of $\overline{BC}$, R is the mid-point of $\overline{CD}$, S is the mid-point of $\overline{DA}$.

P is joined to Q, Q is joined to R. R is joined to S and S is joined to P.

To prove

PQRS is a parallelogram.

Construction

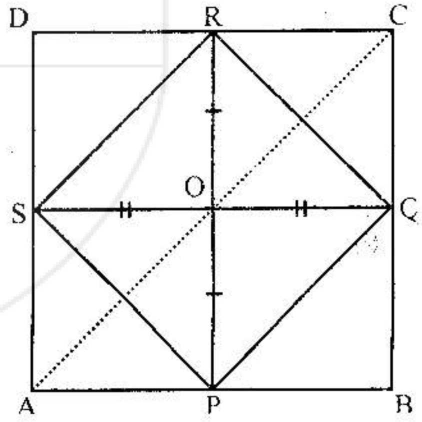
Join A to C.

Proof

Statements	Reasons
In ΔDAC,	
$\left.\begin{array}{l}\overline{SR} \parallel \overline{AC} \\ m\overline{SR} = \frac{1}{2} m\overline{AC}\end{array}\right\}$	S is the mid-point of $\overline{DA}$ R is the mid-point of $\overline{CD}$
In ΔBAC,	
$\left.\begin{array}{l}\overline{PQ} \parallel \overline{AC} \\ m\overline{PQ} = \frac{1}{2} m\overline{AC}\end{array}\right\}$	P is the mid-point of $\overline{AB}$ Q is the mid-point of $\overline{BC}$
$\overline{SR} \parallel \overline{PQ}$	Each $\parallel$ $\overline{AC}$
$m\overline{SR} = m\overline{PQ}$	Each $= \frac{1}{2} m\overline{AC}$
Thus PQRS is a parallelogram	$\overline{SR} \parallel \overline{PQ}$, $m\overline{SR} = m\overline{PQ}$ (proved)

EXERCISE 11.3

(1) Prove that the line-segments joining the mid-points of the opposite sides of a quadrilateral bisect each other.



Given

ABCD is a quadrilateral.

P, Q, R, S are the mid-points of $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ respectively.

P is joined to R, Q is joined to S. $\overline{SQ}, \overline{PR}$ intersect at point "O"

To Prove

$\overline{OP} \cong \overline{OR}, \overline{OS} \cong \overline{OQ}$

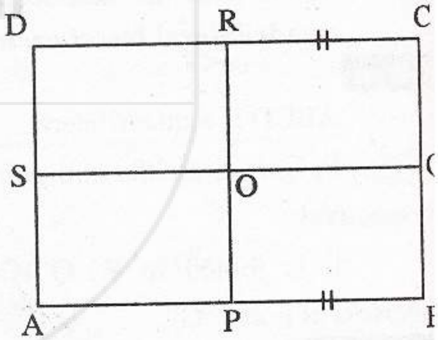
Construction Join P, Q, R, S in order, join A to C.

Proof

Statements	Reasons
$\overline{SR} \parallel \overline{AC}$ (i) $m\overline{SR} = \frac{1}{2} m\overline{AC}$ (ii)	In ΔADC. S, R are mid-points Of $\overline{AD}, \overline{DC}$

And $\overline{PQ} \parallel \overline{AC}$ (iii)	In ΔABC; P, Q are mid-points of $\overline{AB}, \overline{BC}$
$m\overline{PQ} = \frac{1}{2} m\overline{AC}$ (iv)	
$\therefore$ $\overline{PQ} \parallel \overline{SR}$ (v)	from (i), and (iii)
$m\overline{PQ} = m\overline{SR}$ (vi)	From (ii) and (iv)
Similarly $\overline{PS} \parallel \overline{QR}$	
$m\overline{PS} = m\overline{QR}$	
Hence PQRS is a parallelogram	
Now $\overline{PR}$, $\overline{SQ}$ are the diagonals Of PQRS that intersect at point O.	
$\therefore$ $\overline{OP} \cong \overline{OR}$	
$\therefore$ $\overline{OS} \cong \overline{OQ}$	Diagonals of a parallelogram Bisect each other.

(2) Prove that the line-segments joining the mid-points of the opposite sides of a rectangle are the right-bisectors of each other.



Given

ABCD is a rectangle.

and P, Q, R, S are the mid-points of sides $\overline{AB}, \overline{BC}$, $\overline{CD}$ and $\overline{DA}$, respectively.

P is joined to R, S to Q These intersect at "O"

To Prove

$\overline{OQ} \cong \overline{OS}, \overline{OR} \cong \overline{OP}$ and $\overline{RP} \perp \overline{SQ}$

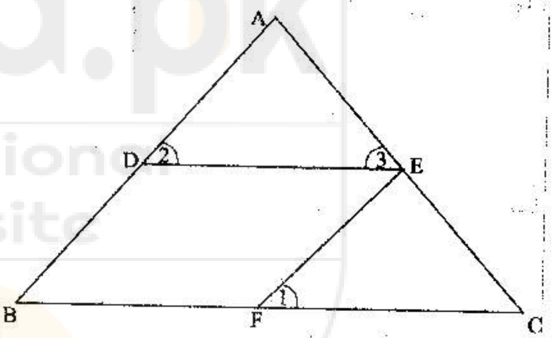
Proof

Statements	Reasons
$AB \parallel CD$	opposite sides of rectangle
$\overline{AP} = \overline{DR}$ (i)	
$m\overline{AB} = m\overline{CD}$	
$\frac{1}{2} m\overline{AB} = \frac{1}{2} m\overline{CD}$	
$m\overline{AP} = m\overline{DR}$ (ii)	
$\therefore$ APRD is rectangle	

$\therefore$ $\overline{OR} \cong \overline{OP}$	As $m\angle A = m\angle D = 90^\circ$
Similarly $\overline{OQ} \cong \overline{OS}$	
Now In rectangle APRD	
$m\overline{DA} = m\overline{RP}$	
$\frac{1}{2} m\overline{DA} = m\overline{RP}$	
$m\overline{DS} = m\overline{RO}$	
$\therefore$ $\overline{DS} \parallel \overline{RO}$,	
Hence SORD is rectangle.	
$\therefore$ $m\angle SOR = 90^\circ$, $\overline{RP} \perp \overline{SQ}$.	

[Note: Diagonals of a rectangle are congruent.]

(9) Prove that the line-segment passing through the mid-point of one side and parallel to another side of a triangle also bisects the third side.



iven In ΔABC, D is mid-point of $\overline{AB}$, $\overline{DE} \parallel BC$ which meets $\overline{AC}$ at E.

equired E is mid-point of $\overline{AB}$ and $\overline{EA} \cong EC$

onstruction

Take $\overline{EF} \parallel \overline{AB}$ which meets $\overline{BC}$ at F.

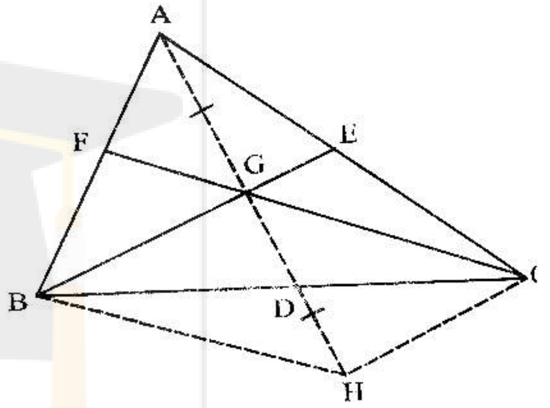
roof

Statements		Reasons
Now BDEF is parallelogram		$\overline{DE} \parallel \overline{BF}$ given, $\overline{EF} \parallel \overline{DB}$ const.
$\therefore$ $\overline{EF} \cong \overline{DB}$	(i)	Opposite sides of parallelogram
$\overline{EF} \cong \overline{AD}$	(ii)	Given
$\angle 1 \cong \angle B$		Corresponding angles.
$\angle 2 \cong \angle B$	(iii)	Corresponding angles.
$\therefore$ $\angle 1 \cong \angle 2$	(iv)	Form (iii)
Now In $\Delta ADE \leftrightarrow \Delta EFC$		
$\angle 1 \cong \angle 2$		Form (iv)
$\angle 3 \cong \angle C$		Corresponding angles.
$\overline{AD} \cong \overline{EF}$		Form (ii)
Hence $\Delta ADE \cong \Delta EFC$		A.A.S $\cong$ A.A.S

$\therefore \quad \overline{AE} \cong \overline{CE}$	Corresponding sides of congruent triangles.

Theorem

The medians of a triangle are concurrent and their point of concurrency is the point of trisection of each median.



Given

ΔABC

To Prove

The medians of the ΔABC are concurrent and the point of concurrency is the point of trisection of each median.

Construction

Draw two medians $\overline{BE}$ and $\overline{CF}$ of the ΔABC which intersect each other at point G. Join A to G and produce it to point H such that $\overline{AG} \cong \overline{GH}$. Join H to the points B and C.

$\overline{AH}$ Intersects $\overline{BC}$ at the point D.

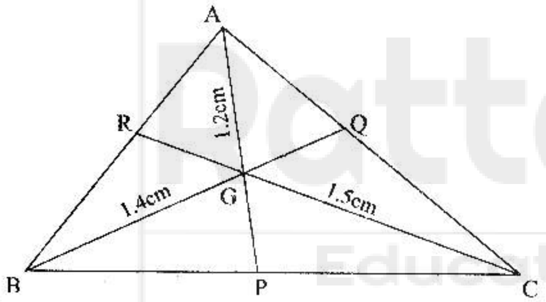
Proof

Statements	Reasons
In ΔACH,	
$\overline{GE} \parallel \overline{HC}$,	G and E are mid-points of sides $\overline{AH}$ and $\overline{AC}$ respectively
or $\overline{BE} \parallel \overline{HC}$ .....(i)	G is a point of $\overline{BE}$
Similarly $\overline{CF} \parallel \overline{HB}$ .....(ii)	
$\therefore$ BHCG is a parallelogram	from (i) and (ii)
and $m\overline{GD} = \frac{1}{2} m\overline{GH}$ .....(iii) $\overline{BD} \cong \overline{CD}$ $\overline{AD}$ is a median of ΔABC	(Diagonals $\overline{BC}$ and $\overline{GH}$ of a parallelogram BHCG intersect each other at point D).
Medians $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ pass through the point G	(G is the intersecting point of $\overline{BE}$ and $\overline{CF}$ and $\overline{AD}$ pass through it.)
Now $\overline{GH} \cong \overline{AG}$ .....(iv)	Construction

$\therefore \quad m\overline{GD} = \frac{1}{2} m\overline{AG}$ and G is the point of trisection of $\overline{AD}$ –(v) similarly it can be proved that G is also the point of trisection of $\overline{CF}$ and $\overline{BE}$.	from (iii) and (iv)

EXERCISE 11.4

(1) The distances of the point of concurrency of the medians of a triangle from its vertices are respectively 1.2cm; 1.4 cm and 1.5 cm. Find the lengths of its medians.



Solution: Let ABC be a triangle with center of gravity at G where $m\overline{AG}=1.2\text{cm}$, $\overline{BG}=1.4\text{cm}$, $m\overline{CG}=1.5\text{cm}$

Required To find the length of AP, BQ, CR

Proof:

$$m\overline{AP} = \frac{3}{2}\times(m\text{AG})$$

$$= \frac{3}{2}\times 1.2 = 1.8\text{cm}$$

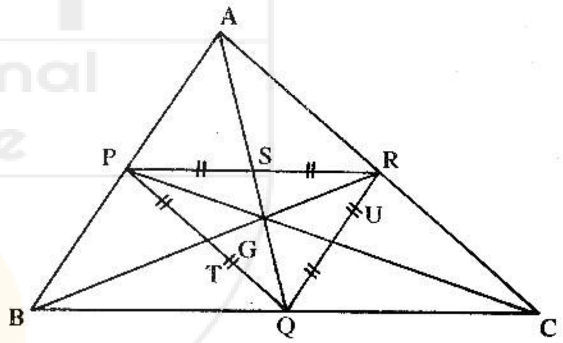
$$m\overline{BQ} = \frac{3}{2}\times(m\overline{BG})$$

$$= \frac{3}{2}\times 1.4 = 2.1\text{cm}$$

$$m\overline{CR} = \frac{3}{2}\times(m\text{CG})$$

$$= \frac{3}{2}\times 1.5 = 2.25\text{cm}$$

(2) Prove that the point of concurrency of the medians of a triangle and the triangle which is made by joining the mid-points of its sides is the same.



Given

In ΔABC, $\overline{AQ}, \overline{BR}, \overline{CP}$ are its medians that are concurrent at point G.

ΔPQR is formed by joining mid-points of $\overline{AB}$, $\overline{BC}$, $\overline{CA}$

To Prove

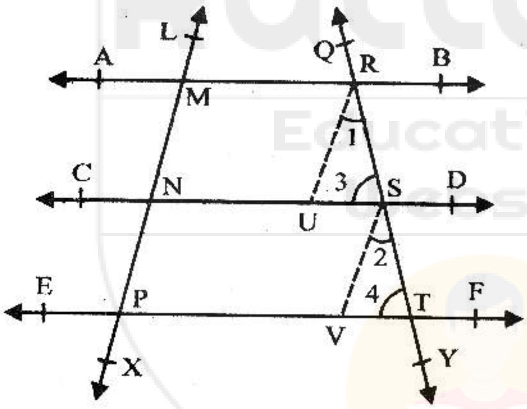
Point G is point of concurrency of triangle PQR.

Proof

Statements	Reasons
$\overline{PR} \parallel \overline{BC}$	P, R are mid-points of $\overline{AB}$ and $\overline{AC}$
$\Rightarrow$ $\overline{PR} \parallel \overline{BQ}$ (i)	
$\overline{RQ} \parallel \overline{AB}$	P, Q are mid-points of $\overline{AB}$ and $\overline{BC}$
$\Rightarrow$ $\overline{RQ} \parallel \overline{PB}$ (ii)	
$\therefore$ PBQR is a parallelogram.	
$\overline{BR}$, $\overline{PQ}$ are its diagonals, that bisect each other at T. T is mid-point $\overline{PQ}$, similarly S is mid-point of $\overline{PR}$ and U is mid-point of $\overline{PQ}$.	

Theorem

If three or more parallel lines make congruent segments on a transversal, they also intercept congruent segments on any other line that cuts them.



Given

$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD} \parallel \overleftrightarrow{EF}$

The transversal $\overleftrightarrow{LX}$ intersects $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ at the points M, N and P respectively, such that $\overline{MN} \cong \overline{NP}$. The transversal $\overleftrightarrow{QY}$ intersects them at points R, S and T respectively.

To Prove

$\overline{RS} \cong \overline{ST}$

Construction

From R, draw $\overline{RU} \parallel \overline{LX}$, which meets $\overline{CD}$ at U. From S, draw $\overline{SV} \parallel \overline{LX}$ which meets $\overline{EF}$ at V. as shown in the figure let the angles be labeled as

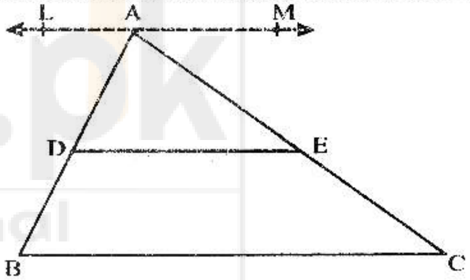
$\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$

Proof

Statements	Reasons
MNUR is a parallelogram	$\overline{RU} \parallel \overline{LX}$ (construction) $\overline{AB} \parallel \overline{CD}$ (given)
$\therefore$ $\overline{MN} \cong \overline{RU}$ .....(i)	(opposite sides of a parallelogram)

Similarly, $\overline{NP} \cong \overline{SV}$ .....(ii) But $\overline{MN} \cong \overline{NP}$ .....(iii) $\therefore$ $\overline{RU} \cong \overline{SV}$	Given {from (i), (ii) and (iii)} Each is $\parallel \overleftrightarrow{LX}$ (construction)
Also $\overline{RU} \parallel \overline{SV}$ $\therefore$ $\angle 1 \cong \angle 2$ and $\angle 3 \cong \angle 4$ In $\Delta RUS \leftrightarrow \Delta SVT$, $\overline{RU} \cong \overline{SV}$ $\angle 1 \cong \angle 2$ $\angle 3 \cong \angle 4$ $\therefore$ $\Delta RUS \cong \Delta SVT$ Hence $\overline{RS} \cong \overline{ST}$	Corresponding angles Corresponding angles Proved Proved Proved S.A.A. $\cong$ S.A.A. (corresponding sides of a congruent triangles)

Corollaries (i) A line, through the mid-point of one side, parallel to another side of a triangle, bisects the third side.



Given In ΔABC, D is the mid-point of $\overline{AB}$. $\overline{DE} \parallel \overline{BC}$ which cuts $\overline{AC}$ at E.

To prove

$\overline{AE} \cong \overline{EC}$

Construction

Through A, draw $\overleftrightarrow{LM} \parallel \overline{BC}$.

Proof

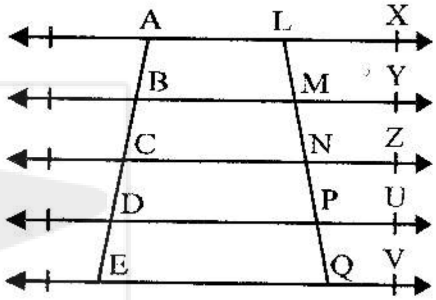
Statements	Reasons
Intercepts cut by $\overleftrightarrow{LM}$, $\overleftrightarrow{DE}$, $\overleftrightarrow{BC}$ on $\overleftrightarrow{AC}$ are congruent. i.e., $\overline{AC} \cong \overline{EC}$	Intercepts cut by parallels $\overleftrightarrow{LM}$, $\overline{DE}$, $\overline{BC}$ on $\overline{AB}$ are congruent (given)

(ii) The parallel line from the mid-point of one non-parallel side of a trapezium to the parallel sides bisects the other non-parallel side.

(iii) If one side of a triangle is divided into congruent segments, the line drawn from the point of division parallel to the other side will make congruent segments on third side.

Exercise 11.5

1. In the given figure. $\overleftrightarrow{AX} \parallel \overleftrightarrow{BY} \parallel \overleftrightarrow{CZ} \parallel \overleftrightarrow{DU} \parallel \overleftrightarrow{EV}$ and $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DE}$ if $m\overline{MN} = 1cm$ then find the length of $\overline{LN}$ and $\overline{LQ}$



Given

In given figure $\overleftrightarrow{AX} \parallel \overleftrightarrow{BY} \parallel \overleftrightarrow{CZ} \parallel \overleftrightarrow{DU} \parallel \overleftrightarrow{EV}$,

$\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DE}$, $m\overline{MN} = 1cm$

Required:

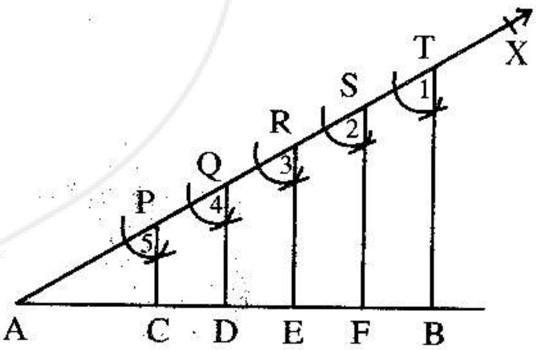
To find $m\overline{LN}$ and $m\overline{LQ}$

Statement	Reasons
$\overline{AX} \parallel \overline{BY} \parallel \overline{CZ} \parallel \overline{DU} \parallel \overline{EV}$	Given
$\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DE}$	Given
$\overline{BC} \cong \overline{MN}$ $\overline{NP} \cong \overline{PQ}$	$\because \parallel$ lines through A, B, C, D, E cut $\overline{LQ}$ in points L, M, N, P, Q.
$m\overline{MN} = 1cm$	Given
$\overline{LN} = 2\overline{MN}$ $= 2(1)$ $= 2cm$	$\because \overline{MN} = 1cm$
$\overline{LQ} = 4\overline{MN}$ $= 4 \times 1$ $= 4cm$	

2. Take a line segment of length 5cm and divide it into five congruent parts.

[Hint: Draw an acute angle $\angle BAX$. On $\overline{AX}$ take $\overline{AP} \cong \overline{PQ} \cong \overline{QR} \cong \overline{RS} \cong \overline{ST}$.

Joint T to B. Draw line parallel to $\overline{TB}$ from the points P, Q, R and S.]



Construction:

(i) Take a line segment AB of 5cm long.

(ii) Draw an acute angle $\angle BAX$.

(iii) Mark 5 points on $\overrightarrow{AX}$ at equal distance starting from point A.

(iv) Join the last point (mark)T to B.

(v) Draw $\overline{SF}, \overline{RE}, \overline{QD}, \overline{PC}$ parallel to TB these line segments meet AB at F,E,D,C points.

Result: AB has been divided into five equal points

$\overline{AC} \cong \overline{CD} \cong \overline{DE} \cong \overline{FB}$

3. Fill in the blanks.

(i) In a parallelogram opposite sides are….. (Parallel / Congruent)

(ii) In a parallelogram opposite angles are ……. (Equal / Congruent)

(iii) Diagonals of a parallelogram …….. each other at a point. (Intersect)

(iv) Medians of a triangle are ………… (Concurrent)

(v) Diagonal of a parallelogram divides the parallelogram into two ……….. triangles. (Congruent)

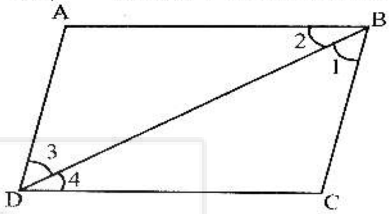
4. In parallelogram ABCD

(i) $m\overline{AB}$ … ≅ ….. $m\overline{DC}$

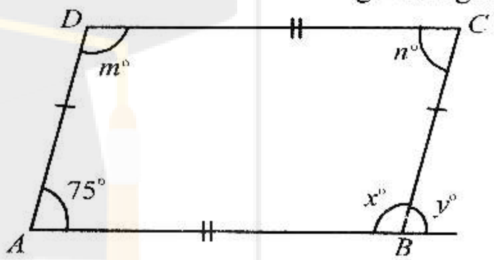
(ii) $m\overline{BC}$ … ≅ … $m\overline{AD}$

(iii) $m\angle 1 \cong$ …$m\angle 3$……

(iv) $m\angle 2 \cong$ …$m\angle 4$……



5. Find the unknowns in the given figure.



Given: Let ABCD be the given figure with

$\overline{AB} \cong \overline{CD}$

$\overline{BC} \cong \overline{AD}$

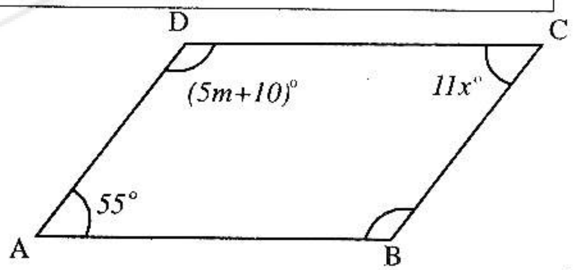
To Find: m°, n°, x°, y°

Proof:

Statement	Reasons
ABCD is a Parallelogram	$\overline{AB} \cong \overline{CD}$ $\overline{AD} \cong \overline{BC}$
$\angle n = 75^\circ$	Opposite interior angles
$m^\circ + 75^\circ = 180^\circ$	supplementary angles
$m^\circ = 180^\circ - 75^\circ = 105^\circ$	
$x^\circ = m^\circ$	
$x^\circ = 105^\circ$	
$x^\circ + y^\circ = 180^\circ$	supplementary angles
$y^\circ = 180^\circ - x^\circ$	
$y^\circ = 180^\circ - 105^\circ$	
$y^\circ = 75^\circ$	

6. If the given figure ABCD is a parallelogram, then find *x*, *m*.

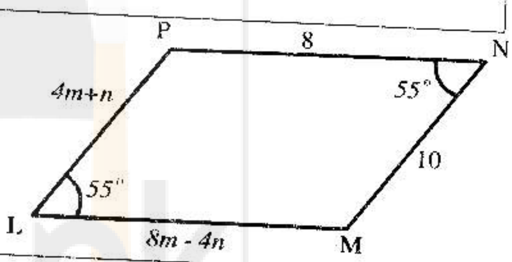
Given: ABCD is a parallelogram with angles as shown To Find x° and m°



Proof:

Statement	Reasons
$11x^\circ = 55^\circ$	Opposite angles of parallelogram
$x^\circ = \frac{55^\circ}{11} = 5^\circ$	
$x^\circ = 5^\circ$	
$(5m + 10)^\circ + 55^\circ = 180^\circ$	Int. supplementary angles
$(5m + 10)^\circ = 180^\circ - 55^\circ$	
$5m^\circ + 10^\circ = 125^\circ$	
$5m^\circ = 125^\circ - 10^\circ$	
$5m^\circ = 115^\circ$	
$m^\circ = 23^\circ$	

7. The given figure LMNP is a parallelogram. Find the value of *m, n.*



Given: The parallelogram LMNP with lengths and angles as shown to find: m° and n°

Proof:

Statement	Reasons
$4m + n = 10$.......(i)	Opposite sides of ‖gm
$8m - 4n = 8$ ....(ii)	Opposite side of ‖gm
Multiplying (i) by 4	
$16m + 4n = 40$ (iii)	
Adding (i) and (iii)	

$$\begin{array}{r} 8m - \cancel{4n} = 8 \\ 16m + \cancel{4n} = 40 \\ \hline 24m = 48 \end{array}$$

$$m = \frac{48}{24} = 2$$

Put in (i)

$4(2) + n = 10$

$8 + n = 10$

$n = 10 - 8 \Rightarrow n = 2$

8. In the question 7, sum of the opposite angles of the parallelogram is 110°, find the remaining angles.

Given: LMNP is a parallelogram with angles 55°, 55° as shown

To Find: All angles

Proof:

Statement	Reasons
$\angle LPN + 55^\circ = 180^\circ$	Interior angles
$\angle LPN = 125^\circ$	
Also	
$\angle m = \angle P$	Opposite angles
$\angle m = 125^\circ$	$\because \angle P = 125^\circ$

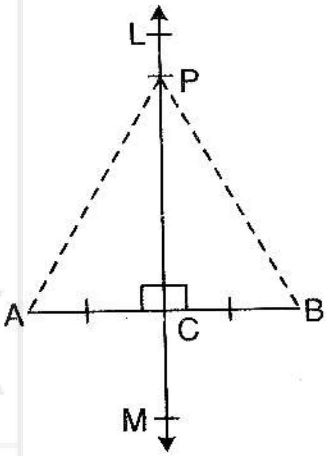
Unit 12 LINE BISECTORS AND ANGLE BISECTORS

Right Bisector of a Line Segment:

A line ℓ is called a right bisector of a line segment if ℓ is perpendicular to the line segment and passes through its mid-point.

Bisector of an Angle:

A ray BP is called the bisector of $\angle ABC$ if P is a point in the interior of the angle and $\angle ABP = \angle PBC$.



Theorem:

Any point on the right bisector of a line segment is equidistant from its end points.

Given:

A line LM intersects the line segment AB at the point C such that $\overleftrightarrow{LM} \perp \overline{AB}$ and $\overline{AC} \cong \overline{BC}$. P is a point on $\overleftrightarrow{LM}$.

To Prove: $\overline{PA} \cong \overline{PB}$

Construction:

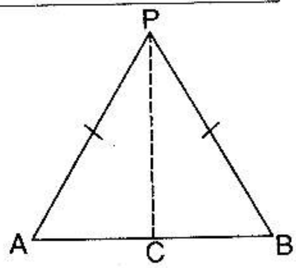
Join p to the points A and B.

Proof

Statements	Reasons
In $\Delta ACP \longleftrightarrow \Delta BCP$	
$\overline{AC} \cong \overline{BC}$	Given
$\angle ACP \cong \angle BCP$	given $\overline{PC} \perp \overline{AB}$, so that each $\angle$ at C = 90°.
$\overline{PC} \cong \overline{PC}$	common
$\therefore$ $\Delta ACP \cong \Delta BCP$	S.A.S. postulate
Hence $\overline{PA} \cong \overline{PB}$	(corresponding sides of congruent triangles)

Theorem

Any point equidistant from the end points of a line segment is on the right bisector of it.



Given

$\overline{AB}$ is a line segment. Point P is such that $\overline{PA} \cong \overline{PB}$.

To Prove The Point P is on the right bisector of $\overline{AB}$.

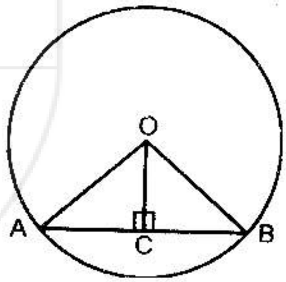
Construction: Joint P to C, the midpoint of $\overline{AB}$.

Proof

Statements		Reasons
In	$\Delta ACP \longleftrightarrow \Delta BCP$	
	$\overline{PA} \cong \overline{PB}$	Given
	$\overline{PC} \cong \overline{PC}$	Common
	$\overline{AC} \cong \overline{BC}$	Construction
	$\Delta ACP \cong \Delta BCP$	S.S.S $\cong$ S.S.S
	$\angle ACP \cong \angle BCP$ .....(i)	(corresponding angles of congruent triangles)
But	$m\angle ACP + m\angle BCP = 180^\circ$ .....(ii)	Supplementary angles
$\therefore$	$m\angle ACP = m\angle BCP = 90^\circ$	From (i) and (ii)
i.e.,	$\overline{PC} \perp \overline{AB}$ .....(iii)	$m\angle ACP = 90^\circ$ (proved)
Also	$\overline{CA} \cong \overline{CB}$ .....(iv)	
$\therefore$	$\overline{PC}$ is a right bisector of $\overline{AB}$.	construction
i.e.,	the point P is on the right bisector of $\overline{AB}$.	from (iii) and (iv)

Exercise 12.1

1. Prove that the centre of a circle is on the right bisectors of each of its chords.



Given Circle with centre O

To Prove Centre of the circle is on right bisectors of each of its chords

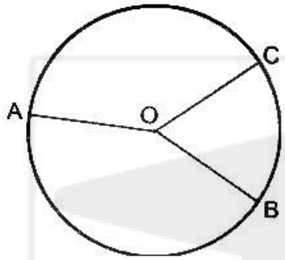
Construction

Draw any chord $\overline{AB}$. Draw $\overline{OC} \perp \overline{AB}$ join O with A and B.

Proof:

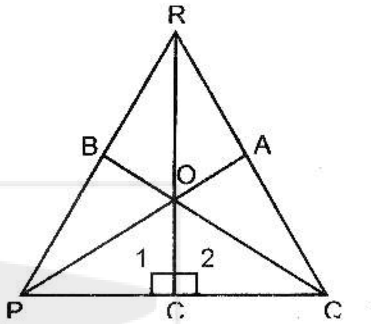
Statements	Reasons
In $\Delta OAC \leftrightarrow \Delta OBC$	
$\overline{OA} \cong \overline{OB}$	Radii of same circle
$\overline{OC} \cong \overline{OC}$	Common
$\angle ACO \cong \angle BCO$	Each of 90°
$\therefore \Delta ACO \cong \Delta BCO$	H.S $\cong$ H.S
$\therefore \overline{AC} \cong \overline{BC}$	Corresponding sides of the congruent triangles.
$\therefore \overline{OC}$ is the right bisector of $\overline{AB}$	

2. Where will be the centre of a circle passing through three non-collinear points and why?



Circle is the locus of a point which moves so that its distance from a fixed point O remains same. Otherwise no circle will be formed.

3. Three villages P, Q and R are not on the same line. The people of these villages want to make a Children Park at such a place which is equidistant from these three villages. After fixing the place, of Children park, prove that the Park is equidistant from the three villages.



Given

Three villages P, Q, R not on the same line.

To Prove

Park is equidistant from P, Q and R.

Construction

Complete the triangle PQR, draw the right bisectors of the sides $\overline{PQ}$ and $\overline{QR}$ cutting each other at O. Join O with P, Q and R. let O be the park.

Proof:

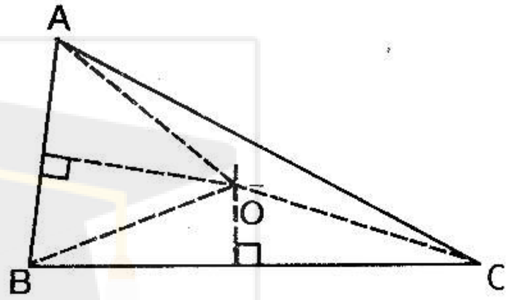
Statements	Reasons
In $\Delta OPC \leftrightarrow \Delta OQC$	
$\overline{CP} \cong \overline{CQ}$	Construction
$\overline{OC} \cong \overline{OC}$	Common
$\angle 1 \cong \angle 2$	Each of 90°
$\therefore$ $\Delta OCP \cong \Delta OCQ$	S.A.S $\cong$ S.A.S
$\therefore$ $\overline{OP} \cong \overline{OQ}$ .... (i)	Corresponding sides of congruent triangles
Similarly	
$\overline{OQ} \cong \overline{OR}$ .... (ii)	
$\therefore$ $\overline{OP} \cong \overline{OQ} \cong \overline{OR}$	

Theorem

The right bisectors of the sides of a triangle are concurrent.

Given

ΔABC



To Prove

The right bisectors of $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ are concurrent.

Construction Draw the right bisectors of $\overline{AB}$ and $\overline{BC}$ which meet each other at the point O. Join O to A, B and C.

Proof:

Statements	Reasons
$\overline{OA} \cong \overline{OB}$ ......(i)	(Each point on right bisector of a segment is equidistant from its end points)
$\overline{OB} \cong \overline{OC}$ ......(ii)	as in (i)
$\overline{OA} \cong \overline{OC}$ ......(iii)	From (i) and (ii)
$\therefore$ Point O is on the right bisector of $\overline{CA}$. ......(iv)	(O is equidistant from A and C)
But point O is on the right bisector of $\overline{AB}$ and of $\overline{BC}$. ......(v)	construction
Hence the right bisectors of the three sides of a triangle are concurrent at O.	{from (iv) and (v)}

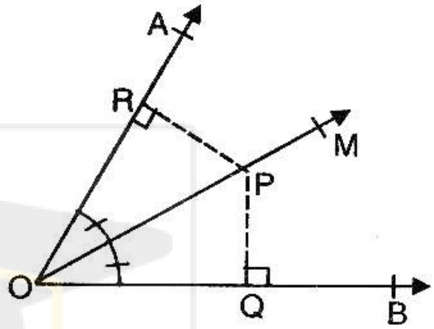
Note:

(a) The right bisectors of the sides of an acute triangle intersect each other inside the triangle.

(b) The right bisectors of the sides of a right triangle intersect each other on the hypotenuse.

(c) The right bisectors of the sides of an obtuse triangle intersect each other outside the triangle.

Theorem



Any point on the bisector of an angle is equidistant from its arms.

Given

A point P is on $\overrightarrow{OM}$, the bisectors of $\angle AOB$.

To Prove

$\overline{PQ} \cong \overline{PR}$ i.e., P is equidistant from $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

Construction

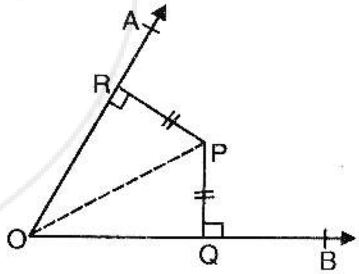
Draw $\overline{PR} \perp \overrightarrow{OA}$ and $\overline{PQ} \perp \overrightarrow{OB}$.

Proof:

Statements	Reasons
In $\Delta POQ \longleftrightarrow \Delta POR$	
$\overline{OP} \cong \overline{OP}$	Common
$\angle PQO \cong \angle PRO$	Construction
$\angle POQ \cong \angle POR$	Given
$\therefore$ $\Delta POQ \cong \Delta POR$	S.A.A. $\cong$ S.A.A.
Hence $\overline{PQ} \cong \overline{PR}$	(corresponding sides of congruent triangles)

Theorem Any point inside an angle, equidistant from its arms, is on the bisector of it.

Given



Any point P lies inside $\angle AOB$ such that $\overline{PQ} \cong \overline{PR}$, where $\overline{PQ} \perp \overrightarrow{OB}$ and $\overline{PR} \perp \overrightarrow{OA}$.

To Prove Point P is on the bisector of $\angle AOB$.

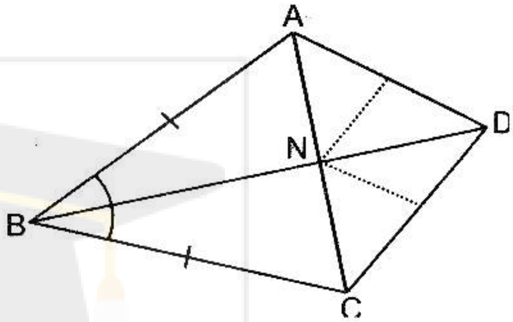
Construction Join P to O.

Proof:

Statements	Reasons
In $\Delta POQ \longleftrightarrow \Delta POR$	
$\angle PQO \cong \angle PRO$	Given (right angles)
$\overline{PO} \cong \overline{PO}$	Common
$\overline{PQ} \cong \overline{PR}$	Given
$\therefore$ $\Delta POQ \cong \Delta POR$	H.S. $\cong$ H.S.
Hence $\angle POQ \cong \angle POR$	(corresponding angles of congruent triangles)
i.e., P is on the bisector of $\angle AOB$.	

Exercise 12.2

1. In a quadrilateral ABCD, $\overline{AB} \cong \overline{BC}$ and the right bisectors of $\overline{AD}$, $\overline{CD}$ meet each other at point N. prove that $\overline{BN}$ is a bisector of $\angle ABC$.



Given Quadrilateral ABCD in which $\overline{AB} \cong \overline{BC}$. Right bisectors of $\overline{AD}$ and $\overline{CD}$ meet each other at point N.

To prove $\overline{BN}$ is a bisector of $\angle ABC$

Construction Join N with A, B, C, D

Proof:

Statements	Reasons
$\overline{NC} \cong \overline{ND}$ .... (i)	N is on the right bisector of $\overline{CD}$
$\overline{NA} \cong \overline{ND}$ .... (ii)	N is on the right bisector of $\overline{AD}$
$\overline{NA} \cong \overline{NC}$ .... (iii)	By (i) and (ii)
In $\Delta ABN \leftrightarrow \Delta CBN$	
$\overline{AB} \cong \overline{BC}$	Given
$\overline{BN} \cong \overline{BN}$	Common
$\overline{NA} \cong \overline{NC}$	Proved
$\therefore$ $\Delta ABN \cong \Delta CBN$	S.S.S $\cong$ S.S.S
$\angle ABN \cong \angle CBN$	Corresponding angles of congruent triangles.
$\therefore$ BN is a bisector of $\angle ABC$.	

2. The bisectors of $\angle A$, $\angle B$ and $\angle C$ of a quadrilateral ABCP meet each other at point O. Prove that the bisectors of $\angle P$ will also pass through the point O.

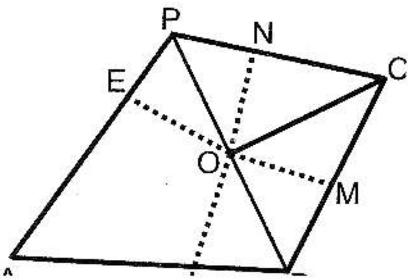
Given Bisector of the angles A, B, C meet at O.

To Prove

Bisector of $\angle P$ will also pass through O.

Construction

From O draw $\perp$ on the sides of quadrilateral BCP.

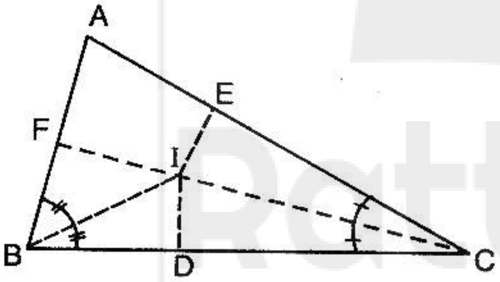


Proof:

Statements	Reasons
$\overline{OE} \cong \overline{OL}$ .... (i)	O is on the bisector of $\angle A$
$\overline{OL} \cong \overline{OM}$ .... (ii)	O is on the bisector of $\angle B$
$\overline{OM} \cong \overline{ON}$ .... (iii)	O is on the bisector of $\angle C$
$\therefore$ $\overline{OE} \cong \overline{ON}$	By (i) and (ii), (iii)
$\therefore$ O is on the bisector of $\angle P$.	$\overline{OE} \cong \overline{ON}$

Theorem

The bisectors of the angles of a triangle are concurrent.



Given

ΔABC

To Prove

The bisectors of $\angle A$, $\angle B$ and $\angle C$ are concurrent.

Construction

Draw the bisectors of $\angle B$ and $\angle C$ which intersect at point I. From I, draw $\overline{IF} \perp \overline{AB}$, $\overline{ID} \perp \overline{BC}$ and $\overline{IE} \perp \overline{CA}$.

Proof:

Statements	Reasons
$\overline{ID} \cong \overline{IF}$	(Any point on bisector of an angle is equidistant from its arms)
Similarly, $\overline{ID} \cong \overline{IE}$	
$\therefore$ $\overline{IE} \cong \overline{IF}$	Each $\cong$ ID, proved.
So, the point I is on the bisector of $\angle A$ ....(i)	
Also the point I is on the bisectors of $\angle ABC$ and $\angle BCA$. ....(ii)	Construction
Thus the bisectors of $\angle A$, $\angle B$ and $\angle C$ are concurrent at I.	{from (i) and (ii)}

Exercise

1. Which of the following are true and which are false?

(i) Bisection means to divide into two equal parts. **(True)**

(ii) Right bisection of line segment means to draw perpendicular which passes through the mid-point. **(True)**

(iii) Any point on the right bisector of a line segment is not equidistant from its end points. **(False)**

(iv) Any point equidistant from the end points of a line segment is on the right bisector of it. **(True)**

(v) The right bisectors of the sides of a triangle are not concurrent. **(False)**

(vi) The bisectors of the angles of a triangle are concurrent. **(True)**

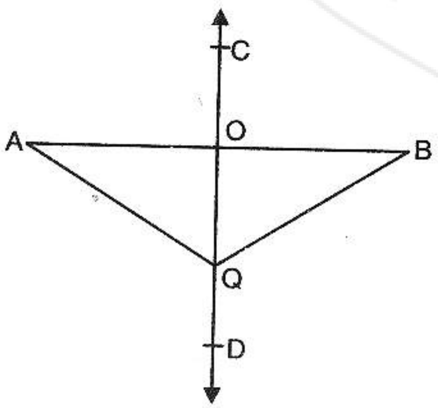
(vii) Any point on the bisector of an angle is not equidistant from its arms **(False)**

(viii) Any point inside an angle, equidistant from its arms, is on the bisector of it. **(True)**

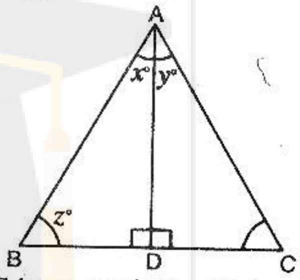
2. If $\overleftrightarrow{CD}$ is right bisector of line segment $\overline{AB}$, then:

(i) $m\overline{OA} =$ ...$m\overline{OB}$...

(ii) $m\overline{AQ} =$ ...$m\overline{BQ}$...



3. The given triangle ABC is equilateral triangle and $\overline{AD}$ is bisector of angle A, then find the values of unknowns x^o, y^o and z^o.



$\because$ ABC is an equilateral triangle.

Its each angle $= 60^o$

$\therefore \quad z = 60^o$

$x + y = 60^o$

But $\quad y = x$

$x + x = 60^o$

$2x = 60^o$

$x = \dfrac{60^o}{2}$

$x = 30^o$

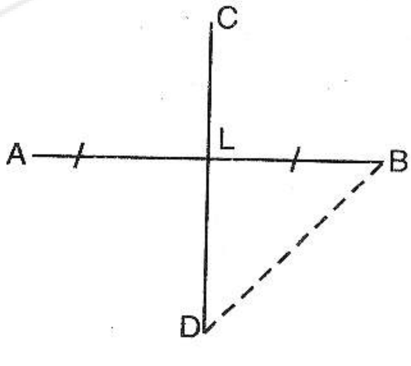
$\therefore \quad y = 30^o$

Hence $\quad z = 60^o$

4. $\overline{CD}$ is right bisector of the line segment $\overline{AB}$.

(i) if $m\overline{AB} = 6\text{cm}$, then find the $m\overline{AL}$ and $m\overline{LB}$.

(ii) If $m\overline{BD} = 4\text{cm}$, then find $m\overline{AD}$.



Given CD is a right bisector on the line segment AB.

To find (i) $m\overline{AL}, m\overline{LB}$ when $m\overline{AB} = 6cm$

(ii) $m\overline{AD}$ when $m\overline{BD} = 4cm$

Construction Join B with D.

Proof:

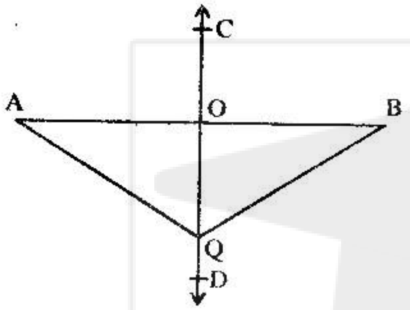
Statements	Reasons
(i) $m\overline{AL} = m\overline{LB}$	$\overline{CD}$ is a right bisector of $\overline{AB}$
$m\overline{AL} = \frac{1}{2} m\overline{AB}$	
$= \frac{1}{2}(6)$	$\because$ $m\overline{AB} = 6cm$
$= 3cm$	
$m\overline{LB} = m\overline{AL}$	
$= 3cm.$	
(ii) $m\overline{AD} = m\overline{BD}$	$\because$ $\overline{LD}$ is a right bisector of $\overline{AB}$
$\therefore$ $m\overline{AD} = 4cm$	$\because$ $m\overline{BD} = 4cm$

Objective

1. Bisection means to divide into ____ equal parts
   (a) Two (b) Three
   (c) Four (d) Five
2. ____ o f line segment means to draw perpendicular which passes through the mid-point of line segment.
   (a) Right bisection (b) Bisection
   (c) Congruent (d) mid-point
3. Any point on the _____ of a line segment is equidistant from its end points:
   (a) Right bisector (b) Angle bisector
   (c) Median (d) Altitude
4. Any point equidistant from the end points of line segment is on the ____ of it:
   (a) Right bisector (b) Angle bisector
   (c) Median (d) Altitude
5. The bisectors of the angles of a triangle are:
   (a) Concurrent (b) Congruent
   (c) Parallel (d) None
6. Bisection of an angle means to draw a ray to divide the given angle into ___ equal parts:
   (a) Four (b) Three
   (c) Two (d) Five
7. If $\overleftrightarrow{CD}$ is right bisector of line segment $\overline{AB}$ then: (i) $m\overline{OA} =$

(a) $m\overline{OQ}$ (b) $m\overline{OB}$

(c) $m\overline{AQ}$ (d) $m\overline{BQ}$



8. If $\overleftrightarrow{CD}$ is right bisector of line segment $\overline{AB}$, then $m\overline{AQ}$ =____

(a) $m\overline{OA}$ (b) $m\overline{OB}$

(c) $m\overline{BQ}$ (d) $m\overline{OD}$

9. The right bisector s of the sides of an acute triangle intersects each other ___ the triangle.

(a) Inside (b) Outside

(c) Midpoint (d) None

10. The right bisectors of the sides of a right triangle intersect each other on the ___

(a) Vertex (b) Midpoint

(c) Hypotenuse (d) None

11. The right bisectors of the sides of an obtuse triangle intersect each other ___ the triangle.

(a) Outside (b) Inside

(c) Midpoint (d) None

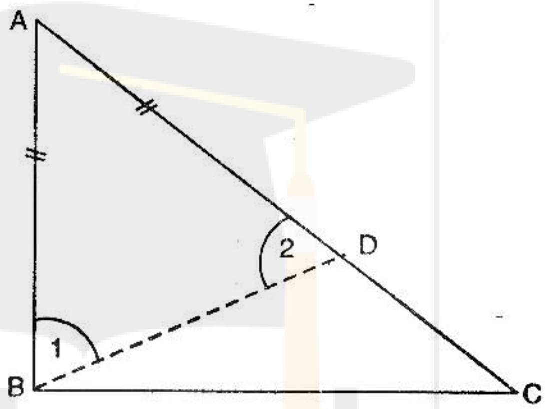
ANSWER KEY

1.	a	**2.**	a	**3.**	a	**4.**	a	**5.**	a
6.	c	**7.**	b	**8.**	c	**9.**	a	**10.**	c
11.	a								

Unit 13

SIDES AND ANGLES OF A TRIANGLE

Theorem If two sides of a triangle are unequal in length, the longer side has an angle of greater measure opposite to it.



Given In ΔABC, $m\overline{AC} > m\overline{AB}$

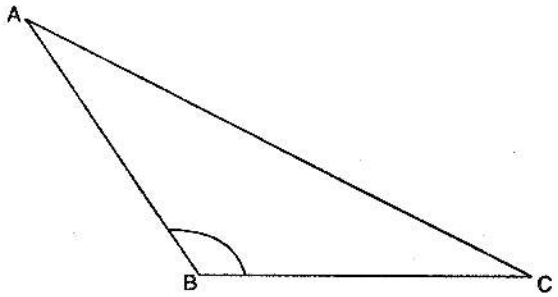
To Prove $m\angle ABC > m\angle ACB$

Construction On $\overline{AC}$ take a point D such that $\overline{AD} \cong \overline{AB}$ Join B to D so that ΔADB is an isosceles triangle. Label $\angle 1$ and $\angle 2$ as shown in the given figure.

Proof

Statements	Reasons
In ΔABD	
$m\angle 1 = m\angle 2$ ...(i)	Angle opposite to congruent sides, (construction)
In ΔBCD, $m\angle ACB < m\angle 2$	
i.e., $m\angle 2 > m\angle ACB$ ...(ii)	(An exterior angle of a triangle is greater than a non-adjacent interior angle).
$\therefore$ $m\angle 1 > m\angle ACB$ ....(iii)	By (i) and (ii)
But	
$m\angle ABC = m\angle 1 + m\angle DBC$	Postulate of addition of angles.
$\therefore$ $m\angle ABC > m\angle 1$ ......(iv)	
$\therefore$ $m\angle ABC > m\angle 1 > m\angle ACB$	
Hence $m\angle ABC > m\angle ACB$	By (iii) and (iv) (Transitive property of inequality of real number)

Example Prove that in a scalene triangle, the angle opposite to the largest side is of measure greater than 60°. (i.e., two-third of a right-angle).



Given In ΔABC, $m\overline{AC} > m\overline{AB}$, $m\overline{AC} > m\overline{BC}$.

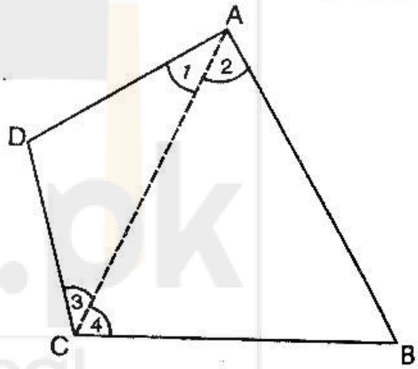
To Prove

$m\angle B > 60^\circ$.

Proof

Statements	Reasons
In ΔABC	
$m\angle B > m\angle C$	$m\overline{AC} > m\overline{AB}$ (given)
$m\angle B > m\angle A$	$m\overline{AC} > m\overline{BC}$ (given)
But $m\angle A + m\angle B + m\angle C = 180^\circ$	$\angle A, \angle B, \angle C$ are the angles of ΔABC
$\therefore$ $m\angle B + m\angle B + m\angle B > 180^\circ$	$m\angle B > m\angle C$, $m\angle B > m\angle A$ (proved)
Hence $m\angle B > 60^\circ$	$180^\circ/3 = 60^\circ$.

Example In a quadrilateral ABCD, $\overline{AB}$ is the longest side and $\overline{CD}$ is the shortest side. Prove that $m\angle BCD > m\angle BAD$.



Given In quad. ABCD, $\overline{AB}$ is the longest side and $\overline{CD}$ is the shortest side.

To Prove $m\angle BCD > m\angle BAD$

Construction

Join A to C.

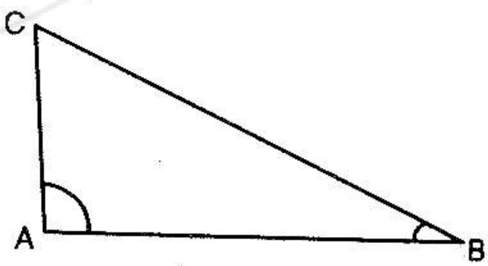
Name the angles $\angle 1$, $\angle 2$, $\angle 3$ and $\angle 4$ as shown in the figure.

Proof

Statements	Reasons
In ΔABC, $m\angle 4 > \angle 2$ ......(i)	$m\overline{AB} > m\overline{BC}$ (given)
In ΔACD, $m\angle 3 > m\angle 1$ ......(ii)	$m\overline{AD} > m\overline{CD}$ (given)
$\therefore$ $m\angle 4 + m\angle 3 > m\angle 2 + m\angle 1$	From I and II
Hence $m\angle BCD > m\angle BAD$	$\begin{cases} m\angle 4 + m\angle 3 = m\angle BCD \\ m\angle 2 + m\angle 1 = m\angle BAD \end{cases}$

Theorem:

If two angles of a triangle are unequal in measure, the side opposite to the greater angle is longer than the side opposite to the smaller angle.



Given In ΔABC, $m\angle A > m\angle B$

To Prove $m\overline{BC} > m\overline{AC}$

Proof

Statements	Reasons
If $m\overline{BC} \not> m\overline{AC}$, then	
either (i) $m\overline{BC} = m\overline{AC}$ or (ii) $m\overline{BC} < m\overline{AC}$	(Trichotomy property of real numbers)
From (i) if $m\overline{BC} = m\overline{AC}$, then $m\angle A = m\angle B$	(Angles opposite to congruent sides are congruent)
which is not possible	Contrary to the given
From (ii) if $m\overline{BC} < m\overline{AC}$, then $m\angle A < m\angle B$	(The angle opposite to longer side is greater than angle opposite to smaller side)
This is also not possible.	Contrary to the given
$\therefore$ $m\overline{BC} \neq m\overline{AC}$	
And $m\overline{BC} \not< m\overline{AC}$	
Thus $m\overline{BC} > m\overline{AC}$	Trichotomy property of real numbers.

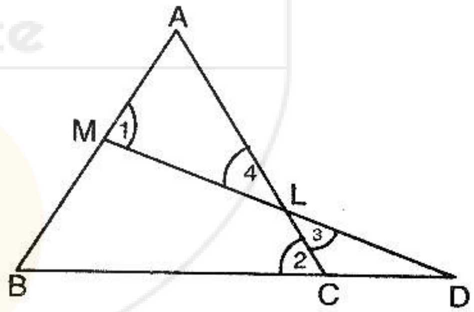
Note

(i) The hypotenuse of a right angle triangle is longer than each of the other two sides.

(ii) In an obtuse angled triangle, the side opposite to the obtuse angle is longer than each of the other two dies.

Example

ABC is an isosceles triangle with base $\overline{BC}$. On $\overrightarrow{BC}$ a point D is taken away from C. A line segment though D cuts $\overline{AC}$ at L and $\overline{AB}$ at M. Prove that $m\overline{AL} > m\overline{AM}$.



Given

In ΔABC, $\overline{AB} \cong \overline{AC}$.

D is a point on $\overrightarrow{BC}$ away from C.

A line segment through D cuts $\overline{AC}$ and L and $\overline{AB}$ at M.

To Prove

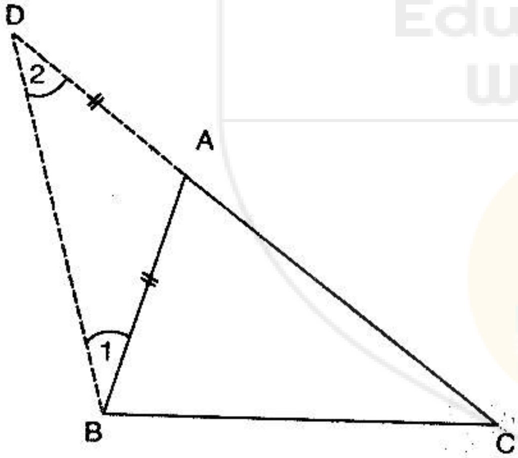
$m\overline{AL} > m\overline{AM}$

Proof

Statements		Reasons
In	ΔABC	
	$\angle B \cong \angle 2$ ...I	$\overline{AB} \cong \overline{AC}$ (given)
In	ΔMBD	
	$m\angle 1 > m\angle B$ ...II	($\angle 1$ is an ext. $\angle$ and $\angle B$ is its internal opposite $\angle$)
$\therefore$	$m\angle 1 > m\angle 2$ ...III	From I and II
In	ΔLCD,	
	$m\angle 2 > m\angle 3$ IV	($\angle 2$ is an ext. $\angle$ and $\angle 3$ is its internal opposite $\angle$)
$\therefore$	$m\angle 1 > m\angle 3$ ...V	From III and IV
But	$\angle 3 \cong \angle 4$ ...VI	Vertical angles
$\therefore$	$m\angle 1 > m\angle 4$	From V and VI
Hence	$m\overline{AL} > m\overline{AM}$	In ΔALM, $m\angle 1 > m\angle 4$ (proved)

Theorem

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.



To Prove

(i) $m\overline{AB} + m\overline{AC} > m\overline{BC}$

(ii) $m\overline{AB} + m\overline{BC} > m\overline{AC}$

(iii) $m\overline{BC} + m\overline{CA} > m\overline{AB}$

Construction

Take a point D on $\overrightarrow{CA}$ such that $\overline{AD} \cong \overline{AB}$. Join B to D and name the angles. $\angle 1$, $\angle 2$ as shown in the given figure.

Given

ΔABC

Proof

Statements		Reasons
In	ΔABD,	
	$\angle 1 \cong \angle 2$ ...(i)	$\overline{AD} \cong \overline{AB}$ (construction)

$m\angle DBC > m\angle 1$ ......(ii)	$m\angle DBC = m\angle 1 + m\angle ABC$
$\therefore$ $m\angle DBC > m\angle 2$ ......(iii)	From (i) and (ii)
In ΔDBC,	
$m\overline{CD} > m\overline{BC}$	By (iii)
i.e., $m\overline{AD} + m\overline{AC} > m\overline{BC}$	$m\overline{CD} = m\overline{AD} + m\overline{AC}$
Hence $m\overline{AB} + m\overline{AC} > m\overline{BC}$	$m\overline{AD} = m\overline{AB}$ (construction)
Similarly,	
$m\overline{AB} + m\overline{BC} > m\overline{AC}$	
And $m\overline{BC} + m\overline{CA} > m\overline{AB}$	

Example

Which of the following sets of lengths can be the lengths of the sides of a triangle.

(a) 2cm, 3cm, 5cm
(b) 3cm, 4cm, 5 cm
(c) 2cm, 4cm, 7cm

(a) $\therefore$ $2+3=5$

$\therefore$ This set of lengths cannot be those of the sides of a triangle.

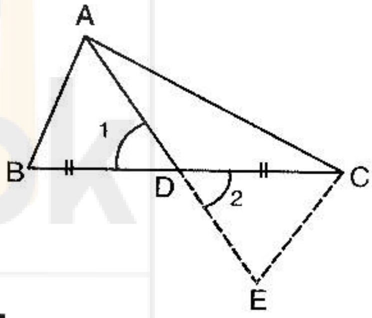
(b) $\therefore$ $3+4>5, 3+5>4, 4+5>3$

$\therefore$ This set can form a triangle.

(c) $\therefore$ $2+4<7$

$\therefore$ This set of lengths cannot be the sides of a triangle.

Example Prove that the sum of the measures of two sides of a triangle is greater than twice the measure of the median which bisects the third side.



Given

In ΔABC,

median AD bisects side $\overline{BC}$ at D.

To Prove

$m\overline{AB} + m\overline{AC} > 2m\overline{AD}$.

Construction On $\overrightarrow{AD}$, take a point E, such that $\overline{DE} \cong \overline{AD}$. Join C to E. Name the angles $\angle 1$, $\angle 2$ as shown in the figure.

Proof

Statements	Reasons
In $\Delta ABD \leftrightarrow \Delta ECD$	
$\overline{BD} \cong \overline{CD}$	Given
$\angle 1 \cong \angle 2$	Vertical angles
$\overline{AD} \cong \overline{ED}$	Construction
$\Delta ABD \cong \Delta ECD$	

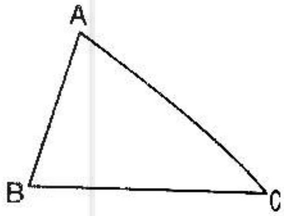
$\overline{AB} \cong \overline{EC}$ I	S.A.S. Postulate
$m\overline{AC} + m\overline{EC} > m\overline{AE}$ II	Corresponding sides of $\cong \Delta s$
$m\overline{AC} + m\overline{AB} > m\overline{AE}$	ACE is a triangle
Hence $m\overline{AC} + m\overline{AB} > 2m\overline{AD}$	From I and II $m\overline{AE} = 2m\overline{AD}$ (construction)

Example

Prove that the difference of measures of two sides of a triangle is less than the measure of the third side.

Given

ΔABC



To Prove

$m\overline{AC} - m\overline{AB} < m\overline{BC}$

$m\overline{BC} - m\overline{AB} < m\overline{AC}$

$m\overline{BC} - m\overline{AC} < m\overline{AB}$

Proof

Statements	Reasons
$m\overline{AB} + m\overline{BC} > m\overline{AC}$	ABC is a triangle
$(m\overline{AB} + m\overline{BC} - m\overline{AB}) > (m\overline{AC} - m\overline{AB})$	Subtracting $m\overline{AB}$ from both sides.
$\therefore$ $m\overline{BC} > (m\overline{AC} - m\overline{AB})$	
Or $m\overline{AC} - m\overline{AB} < m\overline{BC}$ I	$a > b \Rightarrow b < a$
Similarly	
$\left.\begin{array}{l} m\overline{BC} - m\overline{AB} < m\overline{AC} \\ m\overline{BC} - m\overline{AC} < m\overline{AB} \end{array}\right\}$	Reason similar to I

Exercise 13.1

1. Two sides of a triangle measure 10 cm and 15 cm. Which of the following measure is possible for the third side?

(a) 5 cm (b) 20 cm

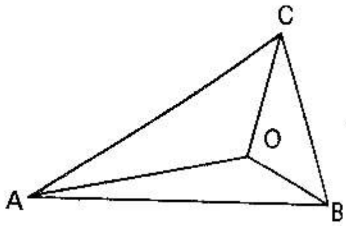
(c) 25 cm (d) 30 cm

Ans. 20cm.

2. O is an interior point of the ABC. Show that

$m\overline{OA} + m\overline{OB} + m\overline{OC} > \frac{1}{2}(m\overline{AB} + m\overline{BC} + m\overline{CA})$

Given: O is the interior point of ΔABC



To Prove:

$$m\overline{OA}+m\overline{OB}+m\overline{OC}>\frac{1}{2}\left(m\overline{AB}+m\overline{BC}+m\overline{CA}\right)$$

Construction:

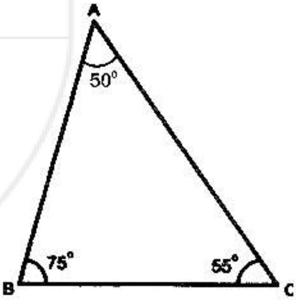
Join O with A, B and C.

Proof:

Statements	Reasons
ΔOAB	
$m\overline{OA}+m\overline{OB}>m\overline{AB}$ ....(i)	Sum of two sides > third side
Similarly	
$m\overline{OB}+m\overline{OC}>m\overline{BC}$ ....(ii)	Sum of two sides > third side
and	
$m\overline{OC}+m\overline{OA}>m\overline{CA}$ ....(iii)	
$2m\overline{OA}+2m\overline{OB}+2m\overline{OC}>m\overline{AB}+m\overline{BC}+m\overline{CA}$	Adding (i), (ii) and (iii)
$2\left(m\overline{OA}+m\overline{OB}+m\overline{OC}\right)>m\overline{AB}+m\overline{BC}+m\overline{CA}$	
$m\overline{OA}+m\overline{OB}+m\overline{OC}>\frac{1}{2}\left(m\overline{AB}+m\overline{BC}+m\overline{CA}\right)$	

3. In the ΔABC, $m\angle B = 75^o$ and $m\angle C = 55^o$. Which of the sides of the triangle is longest and which is the shortest?

Ans: Given a ΔABC in which



$m\angle B = 75^0$

$m\angle C = 55^0$

As $m\angle A + m\angle B + m\angle C = 180^0$

$m\angle A + 75^0 + 55^0 = 180^0$

$m\angle A + 130^0 = 180^0$

$m\angle A = 180^0 - 130^0$

$m\angle A = 50^0$

As we know in a triangle, the side opposite to greater angle is longer than the side opposite to smaller angle

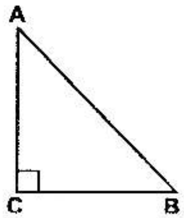
So $m\overline{AC} > m\overline{BC}$

Hence longest side is $\overline{AC}$

and shortest side is $\overline{BC}$

4. Prove that in a right-angled triangle, the hypotenuse is longer than each of the other two sides.

Ans.



Given: ΔABC is a right angle triangle.

Hence AB is hypotenuse of ΔABC.

To prove:

$m\overline{AB} > m\overline{AC}$ and $m\overline{AB} > m\overline{BC}$

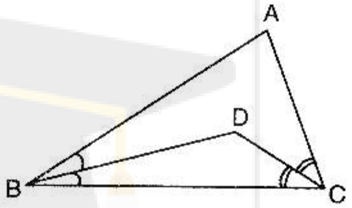
Proof:

As ΔABC is a right angle triangle. So $m\angle C = 90^0$ is the largest angle and the remaining angles $\angle A$ and $\angle B$ are acute. So $m\angle C > m\angle A$ and $m\angle C > m\angle B$

As the side opposite to the greater angle is longer than the side opposite to the smaller angle.

Hence $m\overline{AB} > m\overline{AC}$ and $m\overline{AB} > m\overline{BC}$

5. In the triangular figure, $\overline{AB} > \overline{AC}$. $\overline{BD}$ and $\overline{CD}$ are the bisectors of $\angle B$ and $\angle C$ respectively. Prove that $\overline{BC} > \overline{DC}$.



Given: $\overline{AB} > \overline{BC}$, $\overline{BD}$ and $\overline{CD}$ are the bisectors of the angles B and C

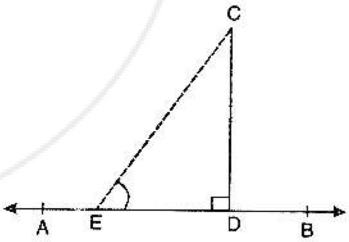
To Prove:

To prove = $\overline{BD} > \overline{CD}$

Proof

Statements	Reasons
$\therefore$ in ΔABC $\angle ACB > \angle ABC$ $\frac{1}{2}\angle ACB > \frac{1}{2}\angle ABC$	$\therefore$ $\overline{AB} > \overline{AC}$
$\therefore$ $\angle BCD > \angle DBC$ $\overline{BD} > \overline{CD}$	$\overline{CD}, \overline{BD}$ are bisectors of $\angle C$, $\angle B$. The bigger sides is opposite the bigger angle

Theorem From a point, outside a line, perpendicular is the shortest distance from the point to the line.



Given A line AB and a point C (not lying on $\overleftrightarrow{AB}$) and a point D on $\overleftrightarrow{AB}$ such that

$\overline{CD} \perp \overleftrightarrow{AB}$.

To Prove

$m\overline{CD}$ is the shortest distance from the point C to $\overleftrightarrow{AB}$.

Construction

Take a point E on $\overleftrightarrow{AB}$. Join C and E to form a ΔCDE

Proof:

Statements	Reasons
In ΔCDE $m\angle CDB > m\angle CED$	(An exterior angle of a triangle is greater

	than non adjacent interior angle).
But $m\angle CDB = m\angle CDE$	Supplement of right angle.
$\therefore$ $m\angle CDE > m\angle CED$	
or $m\angle CED < m\angle CDE$	$a > b \Rightarrow b < a$
or $m\overline{CD} < m\overline{CE}$	
But E is any point on $\overline{AB}$	Side opposite to greater angle is greater.
Hence $m\overline{CD}$ is the shortest distance from C to $\overleftrightarrow{AB}$.	

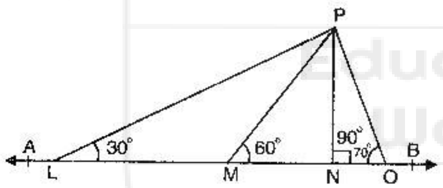
Note:

(i) The distance between a line and a point not on it, is the length of the perpendicular line segment from the point to the line.

(ii) The distance between a line and a point lying on it is zero

Exercise 13.2

1. In the figure, P is any point and AB is a line. Which of the following is the shortest distance between the point P and the line AB.



(a) $m\overline{PL}$ (b) $m\overline{PM}$

(c) $m\overline{PN}$ (d) $m\overline{PO}$

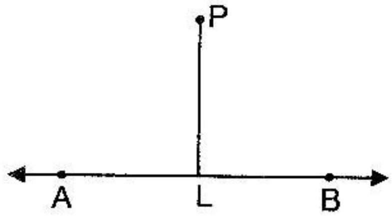
Ans. (c)

2. In the figure, P is any point lying away from the line AB. Then $m\overline{PL}$ will be the shortest distance if:

(a) $m\angle PLA = 80^\circ$

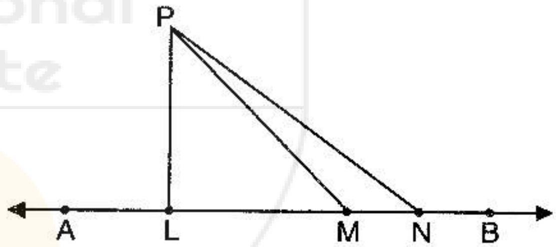
(b) $m\angle PLB = 100^\circ$

(c) $m\angle PLA = 90^\circ$



Ans. (c)

3. In the figure, $\overline{PL}$ is perpendicular to the line AB and $m\overline{LN} > m\overline{LM}$. Prove that $m\overline{PN} > m\overline{PM}$.



Ans. Here it is given $m\overline{PL}$ is perpendicular to line $\overleftrightarrow{AB}$ and $m\overline{LN} > m\overline{LM}$

Proof:

Here $m\overline{PN} > m\overline{PM}$

As $\overline{PL}$ is the shortest distance from P to line $\overleftrightarrow{AB}$. So

$\overline{PL} \perp \overline{AB}$

As we go away from point L, the distance from points to L increases Hence

$m\overline{PN} > m\overline{PM}$

4. Which of the following are true and which are false?

(i) The angle opposite to the longer side is greater. **TRUE**

(ii) In a right-angled triangle greater angle is of 60^o. **FALSE**

(iii) In an isosceles right-angled triangle, angles other than right angle are each of 45^o. **TRUE**

(iv) A triangle having two congruent sides is called equilateral triangle. FALSE

(v) A perpendicular from a point to t line is shortest distance. **TRUE**

(vi) Perpendicular to line form an angle of 90^o. **TRUE**

(vii) A point out-side the line is collinear. **FALSE**

(viii) Sum of two sides of triangle is greater than the third. **TRUE**

(ix) The distance between a line and a point on it is zero. **TRUE**

(x) Triangle can be formed of lengths 2 cm, 3 cm and 5 cm. **FALSE**

5. What will be angle for shortest distance from an outside point to the line?

Ans. 90^o

6. If 13 cm, 12 cm, and 5 cm are the lengths of a triangle, then verify that difference of measures of any two sides of a triangle is less than the measure of the third side.

Ans: (i) $13 - 12 = 1 < 15$

(ii) $12 - 4 = 7 < 13$

(iii) $13 - 5 = 8 < 12$

So verified

7. If 10 cm,6 cm and 8 cm are the lengths of a triangle, then verify that sum of measures of two sides of a triangle is greater than the third side.

Ans. (i) $10 + 6 = = 16 > 8$

(ii) $6 + 8 = 14 > 10$

(iii) $10+8 = 18 > 6$

8. 3 cm, 4 cm and 7 are not the lengths of the triangle. Give the reason.

Ans: $3 + 4 \not> 7$

9. If 3 cm and 4 cm are lengths of two sides of a right angle triangle then what should be the third length of the triangle.

Ans. Third length $= \sqrt{3^2 + 4^2}$

$= \sqrt{25} = 5cm$

OBJECTIVE

1. Which of the following sets of lengths can be the lengths of the sides of a triangle:

(a) 2cm, 3cm, 5cm

(b) 3cm, 4cm, 5cm

(c) 2cm, 4cm, 7cm

(d) None

2. Two sides of a triangle measure 10cm and 15cm. Which of the following measure is possible for the third side!

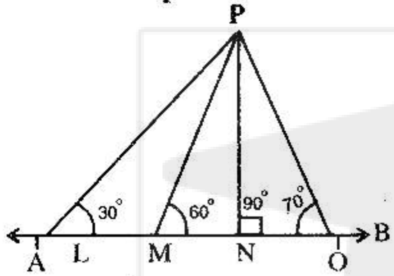
(a) 5cm

(b) 20cm

(c) 25cm

(d) 30cm

3. In the figure, P is any point and AB is a line. Which of the following is the short distance between the point P and line AB.



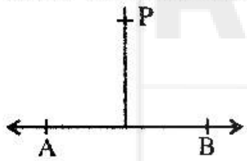
(a) $m\overline{PL}$

(b) $m\overline{PM}$

(c) $m\overline{PN}$

(d) $m\overline{PO}$

4. In the figure, P is any point lying away from the line AB. Then $m\overline{PL}$ will be shortest distance if:



(a) $m < PLA = 80°$

(b) $m < PLB = 100°$

(c) $m < PLA = 90°$

(d) None

5. The angle opposite to the longer side is:

(a) Greater

(b) Shorter

(c) Equal

(d) None

6. In right angle triangle greater angle of:

(a) 60°

(b) 30°

(c) 75°

(d) 90°

7. In an isosceles right-angled triangle angles other than right angle are each of:

(a) 40°

(b) 45°

(c) 50°

(d) 55°

8. A triangle having two congruent sides is called ___ triangle.

(a) Equilateral

(b) Isosceles

(c) Right

(d) None

9. Perpendicular to line form an angle of ___

(a) 30°

(b) 60°

(c) 90°

(d) 120°

10. Sum of two sides of triangle is ___ than the third.

(a) Greater

(b) Smaller

(c) Equal

(d) None

11. The distance between a line and a point on it is ___

(a) Zero

(b) One

(c) Equal

(d) None

ANSWER KEY

1.	b	**2.**	b	**3.**	c	**4.**	c	**5.**	a
6.	d	**7.**	b	**8.**	a	**9.**	c	**10.**	a
11.	a								

Unit 14

RATIO AND PROPORTION

14.1 Ratio and Proportion

We defined ratio $a:b=\frac{a}{b}$ as the comparison of two alike quantities a and b, called the elements (terms) of a ratio. (Elements must be expressed in the same units). Equality of two ratios was defined as proportion.

That is, if $a:b=c:d$, then a,b,c and d are said to be in proportion.

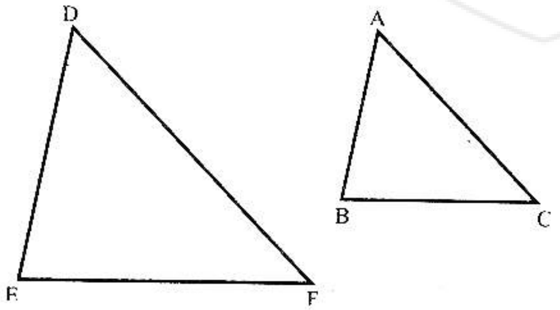
Similar Triangles

Equally important are the similar shapes. In particular the similar triangles that have many practical applications. For example, we know that a photographer can develop prints to different sizes from the same negative. In spite of the difference in size, these pictures look like each other. One photograph is simply an enlargement of another. They are said to be similar in shape. Geometrical figures can also be similar. e.g., if

In $\Delta ABC \longleftrightarrow \Delta DEF$

$\angle A \cong \angle D,\quad \angle B \cong \angle E,\quad \angle C \cong \angle F,$

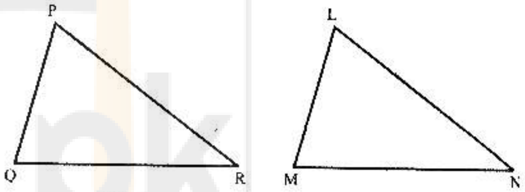
and $\frac{\overline{AB}}{\overline{DE}}=\frac{\overline{BC}}{\overline{EF}}=\frac{\overline{CA}}{\overline{FD}}$



then ΔABC and ΔDEF are called similar triangles which is symbolically written as $\Delta ABC \sim \Delta DEF$

It means that corresponding angles of similar triangles are equal and measures of their corresponding sides are proportional.

$\Delta PQR \cong \Delta LMN$ means that in



$\Delta PQR \longleftrightarrow \Delta LMN$

$\angle P \cong \angle L,\qquad \angle Q \cong \angle M$

$\angle R \cong \angle N,\qquad \overline{PQ} \cong \overline{LM},$

$\overline{QR} \cong \overline{MN},\qquad \overline{RP} \cong \overline{NL},$

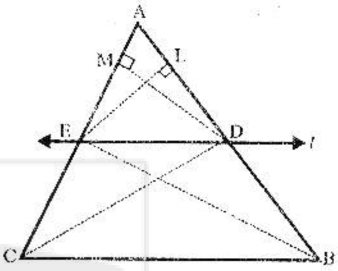
Now as $\frac{\overline{PQ}}{\overline{LM}}=\frac{\overline{QR}}{\overline{MN}}=\frac{\overline{RP}}{\overline{NL}}=1$

$\therefore\quad \Delta PQR \sim \Delta LMN$

Note:

Two congruent triangles are similar also. But two similar triangles are not necessarily congruent, as congruence of their corresponding sides is not necessary.

Theorem A line parallel to one side of a triangle and intersecting the other two sides divides them proportionally.



Given In ΔABC, the line l is intersecting the sides $\overline{AC}$ and $\overline{AB}$ at points E and D respectively such that $\overline{ED} \parallel \overline{CB}$.

To Prove

$m\overline{AD} : m\overline{BD} = m\overline{AE} : m\overline{EC}$

Construction

Join B to E and C to D. From D draw $\overline{DM} \perp \overline{AC}$ and from E draw $\overline{EL} \perp \overline{AB}$.

Proof

Statements	Reasons
In triangles BED and AED, $\overline{EL}$ is the common perpendicular.	
$\therefore\ \Delta BED = \frac{1}{2} \times m\overline{BD} \times m\overline{EL}$..........$(i)$ and $\Delta AED = \frac{1}{2} \times m\overline{AD} \times m\overline{EL}$..........$(ii)$	Area of a $\Delta = \frac{1}{2}$(base) (height)
Thus $\frac{\Delta BED}{\Delta AED} = \frac{m\overline{BD}}{m\overline{AD}}$ ....(iii)	Dividing (i) by (ii)
Similarly $\frac{\Delta CDE}{\Delta ADE} = \frac{m\overline{EC}}{m\overline{AE}}$ ....(iv)	
But $\Delta BED \cong \Delta CDE$	Areas of triangles with common base and same altitudes are equal. Given that $\overline{ED} \parallel \overline{CB}$ so altitudes are equal.
$\therefore$ From (iii) and (iv), we have $\frac{m\overline{BD}}{m\overline{AD}} = \frac{m\overline{EC}}{m\overline{AE}}$ or $\frac{m\overline{AD}}{m\overline{BD}} = \frac{m\overline{AE}}{m\overline{EC}}$	Taking reciprocal of both sides.
Hence $m\overline{AD} : m\overline{BD} = m\overline{AE} : m\overline{EC}$	

Note:

From the above theorem we also have

$$\frac{m\overline{BD}}{m\overline{AB}} = \frac{m\overline{CE}}{m\overline{AC}} \text{ and } \frac{m\overline{AD}}{m\overline{AB}} = \frac{m\overline{AE}}{m\overline{AC}}$$

Corollaries

a) If $\dfrac{m\overline{AD}}{m\overline{AB}}=\dfrac{m\overline{AE}}{m\overline{AC}}$, then $\overline{DE}\parallel\overline{BC}$

b) If $\dfrac{m\overline{AB}}{m\overline{DB}}=\dfrac{m\overline{AC}}{m\overline{EC}}$, then $\overline{DE}\parallel\overline{BC}$

Note:

i) Two points determine a line and three non-collinear points determine a plane.

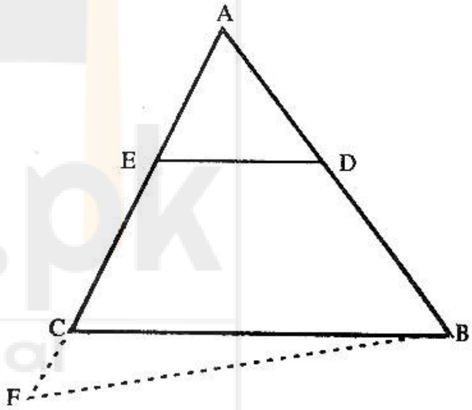
ii) A line segment has exactly one midpoint.

iii) If two intersecting lines from equal adjacent angles, the lines are perpendicular.

Theorem

(Converse of Theorem)

If a line segment intersects the two sides of a triangle in the same ratio then it is parallel to the third side.



Given In ΔABC, $\overline{ED}$ intersects $\overline{AB}$ and $\overline{AC}$ such that $m\overline{AD}:m\overline{BD}=m\overline{AE}:m\overline{EC}$

To Prove $\overline{ED}\parallel\overline{CB}$

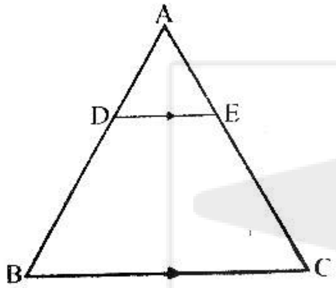
Construction If $\overline{ED}\nparallel\overline{CB}$, then draw $\overline{BF}\parallel\overline{DE}$ to meet $\overline{AC}$ produced at F.

Proof

Statements	Reasons
In ΔABF	
$\overline{DE}\parallel\overline{BF}$	Construction
$\therefore\ \dfrac{m\overline{AD}}{m\overline{DB}}=\dfrac{m\overline{AE}}{m\overline{EF}}$ ....(i)	(A line parallel to one side of a triangle divides the other two sides proportionally)
But $\dfrac{m\overline{AD}}{m\overline{BD}}=\dfrac{m\overline{AE}}{m\overline{EC}}$ ....(ii)	Given
$\therefore\ \dfrac{m\overline{AE}}{m\overline{EF}}=\dfrac{m\overline{AE}}{m\overline{EC}}$	From (i) and (ii)
or $m\overline{EF}=\overline{EC}$	
which is possible only if point F is coincident with C.	(Property of real numbers)
$\therefore$ Our supposition is wrong.	
Hence $\overline{ED}\parallel\overline{CB}$	

Exercise 14.1

1. In ΔABC, $\overline{DE} \parallel \overline{BC}$



i) $\overline{AD} = 1.5\text{ cm}$, $\overline{BD} = 3\text{ cm}$, $\overline{AE} = 1.3\text{cm}$ then find $\overline{CE}$.

ii) If $\overline{AD} = 2.4\text{ cm}$, $\overline{AE} = 3.2\text{ cm}$, $\overline{EC} = 4.8\text{ cm}$, find $\overline{AB}$

iii) If $\dfrac{\overline{AD}}{\overline{DB}} = \dfrac{3}{5}$, $\overline{AC} = 4.8\text{ cm}$, find $\overline{AE}$

iv) If $\overline{AD} = 2.4\text{ cm}$, $\overline{AE} = 3.2\text{ cm}$, $\overline{DE} = 2\text{ cm}$, $\overline{BC} = 5\text{ cm}$, find $\overline{AB}, \overline{DB}, \overline{AC}, \overline{CE}$

v) If $\overline{AD} = 4x-3$, $\overline{AE} = 8x-7$, $\overline{BD} = 3x-1$, and $\overline{CE} = 5x-3$, find the value of x

In ΔABC, $\overline{DE} \parallel \overline{BC}$

(i)

$$\frac{m\overline{AD}}{m\overline{BD}} = \frac{m\overline{AE}}{m\overline{EC}}$$

$$\frac{1.5}{3} = \frac{1.3}{m\overline{EC}}$$

$$m\overline{EC} = \frac{3\times 1.3}{1.5}$$

$$= 2.6\text{cm}$$

(ii) In ΔABC, $\overline{DE} \parallel \overline{BC}$

$$m\overline{AB} = m\overline{AD} + m\overline{BD}$$

Let $m\overline{DB} = x\text{ cm}$

Now $\dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{m\overline{AE}}{m\overline{EC}}$

$$\frac{2.4}{x} = \frac{3.2}{4.8}$$

$$x = \frac{4.8\times 2.4}{3.2}$$

$$x = \frac{48\times 24}{10\times 32}$$

$$x = 3.6\text{cm}.$$

$\therefore$ $m\overline{AB} = m\overline{AD} + m\overline{BD}$

$$m\overline{AB} = 2.4 + 3.6 = 6\text{cm}$$

(iii) $\dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{3}{5}$, $m\overline{AC} = 4.8\text{cm}$

In ΔABC, $\overline{DE} \parallel \overline{BC}$

$$\frac{m\overline{AD}}{m\overline{DB}} = \frac{m\overline{AE}}{m\overline{EC}}$$

$$\frac{m\overline{AD}}{m\overline{DB}} = \frac{m\overline{AC} - m\overline{CE}}{m\overline{CE}}$$

$$\frac{3}{5} = \frac{4.8 - m\overline{CE}}{m\overline{CE}}$$

$$3m\overline{CE} = 5(4.8 - m\overline{CE})$$

$$3m\overline{CE} = 24 - 5m\overline{CE}$$

$$3m\overline{CE} + 5m\overline{CE} = 24$$

$$8m\overline{CE} = 24$$

$$m\overline{CE} = \frac{24}{8} = 3\text{cm}$$

$$m\overline{AE} = m\overline{AC} - m\overline{CE}$$

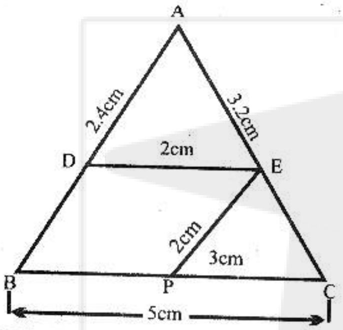
$$= 4.8 - 3$$

$$m\overline{AE} = 1.8\text{cm}$$

(iv) $m\overline{AD} = 2.4cm$,

$m\overline{AE} = 3.2\,cm, m\overline{DE} = 2cm,\; m\overline{BC} = 5cm.$

$m\overline{AB} = ?\;\; m\overline{DB} = ?\; m\overline{AC} = ?\, m\overline{CE} = ?$



$\overline{EP} \parallel \overline{AB}$

DEPB is a parallelogram, then

$m\overline{PB} = mDE = 2cm.$

$m\overline{CP} = 5 - 2 = 3cm$

In $\Delta ABC,\; \overline{EP} \parallel \overline{AB}$

$$\frac{m\overline{CE}}{m\overline{EA}} = \frac{m\overline{CP}}{m\overline{PB}}$$

$$\frac{m\overline{CE}}{3.2} = \frac{3}{2}$$

$$m\overline{CE} = \frac{3\times 3.2}{2}$$

$m\overline{CE} = 3\times 1.6 = 4.8cm$

Now $\overline{DE} \parallel \overline{BC}$, in ΔABC

$$\frac{m\overline{BD}}{m\overline{AD}} = \frac{m\overline{CE}}{m\overline{AE}}$$

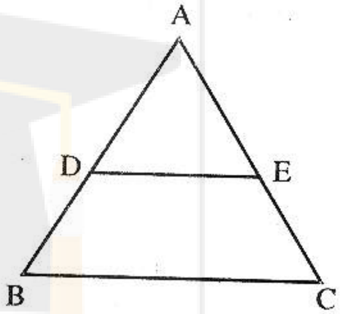
$$\frac{m\overline{BD}}{2.4} = \frac{4.8}{3.2}$$

$$m\overline{BD} = \frac{2.4\times 4.8}{3.2} = 3.6cm$$

$m\overline{AB} = m\overline{AD} + m\overline{DB}$
$= 2.4 + 3.6$
$= 6.0cm$

$m\overline{AC} = m\overline{AE} + m\overline{EC}$
$= 3.2 + 4.8$
$= 8.0cm.$

(v) If $\overline{AD} = 4x - 3, \overline{AE} = 8x - 7, \overline{BD} = 3x - 1$ and $\overline{CE} = 5x - 3$, Find the value of x



In $\Delta ABC, \overline{DE} \parallel \overline{BC}$

$$\frac{m\overline{AD}}{m\overline{BD}} = \frac{m\overline{AE}}{m\overline{CE}}$$

$$\frac{4x-3}{3x-1} = \frac{8x-7}{5x-3}$$

$(4x-3)(5x-3) = (8x-7)(3x-1)$

$20x^2 - 27x + 9 = 24x^2 - 29x + 7$

$20x^2 - 24x^2 - 27x + 29x + 9 - 7 = 0$

$-4x^2 + 2x + 2 = 0$

$2x^2 - x - 1 = 0$

$2x^2 - 2x + x - 1 = 0$

$2x(x-1) + 1(x-1) = 0$

$(x-1)(2x+1) = 0$

$x - 1 = 0$ or $2x + 1 = 0$

$x=1$ or $2x = -1$

$x=1$ or $x = \frac{-1}{2}$

But $x = \frac{-1}{2}$ not possible

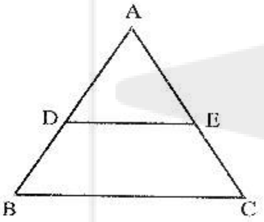
So $x = 1$

2. If ΔABC is an isosceles triangle, $\angle A$ is vertex angle and $\overrightarrow{DE}$ intersects the

sides $\overline{AB}$ and $\overline{AC}$ as shown in the figure so that.

$$m\overline{AD} : m\overline{DB} = m\overline{AE} : m\overline{EC}$$

Prove that ΔADE is also an isosceles triangle.



In ΔABC, $\angle A$ is vertical angle and $\overline{AB} \cong \overline{AC}$

$$\frac{m\overline{AD}}{m\overline{DB}} = \frac{m\overline{AE}}{m\overline{EC}}$$

$$\frac{m\overline{DB}}{m\overline{AD}} = \frac{m\overline{EC}}{m\overline{AE}}$$

$$\frac{m\overline{DB} + m\overline{AD}}{m\overline{AD}} = \frac{m\overline{EC} + m\overline{AE}}{m\overline{AE}}$$

$$\frac{m\overline{AB}}{m\overline{AD}} = \frac{m\overline{AC}}{m\overline{AE}}$$

Now $\quad m\overline{AB} = m\overline{AC}$

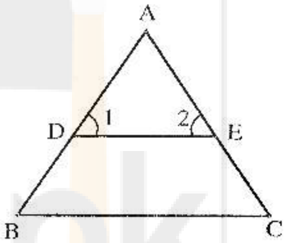
$\quad m\overline{AD} = m\overline{AE}$

Δ ADE is an isosceles triangle.

3. In an equilateral triangle ABC shown in the figure.

$$m\overline{AE} : m\overline{AC} = m\overline{AD} : m\overline{AB}$$

Find all three angles of ΔADE and name it also.



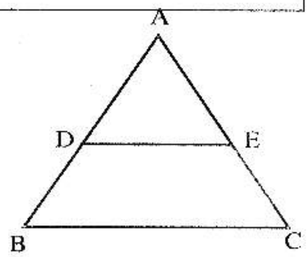
Given: ΔABC is an equilateral triangle.

$$\frac{m\overline{AE}}{m\overline{AC}} = \frac{m\overline{AD}}{m\overline{AB}}$$

To Prove: Find all angles of ΔADE

Statements	Reasons
$\frac{m\overline{AE}}{m\overline{AC}} = \frac{m\overline{AD}}{m\overline{AB}}$	Given
Then $\overline{DE} \parallel \overline{BC}$	Proved
ΔABC is equilateral triangle	Corresponding angle
Then $m\angle A = m\angle B = m\angle C = 60^\circ$	
$\overline{DE} \parallel \overline{BC}$	
$m\angle 1 = m\angle B = 60^\circ$	
$m\angle 2 = m\angle C = 60^\circ$	
$m\angle A = 60^\circ$	

4. Prove that the line segment drawn through the mid-point of one side of a triangle and parallel to another side bisects the third side.



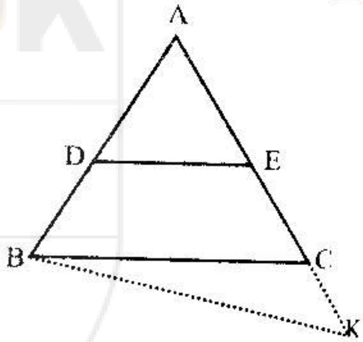
Given in $\Delta ABC, \overline{DE}$ is such that $m\overline{AD} = m\overline{DB}$ and $\overline{DE} \parallel \overline{BC}$

To Prove:

$m\overline{AE} = m\overline{EC}$

Statements	Reasons
In ΔABC	
$\overline{DE} \parallel \overline{BC}$	Given
$\dfrac{m\overline{AD}}{m\overline{BD}} = \dfrac{m\overline{AE}}{m\overline{EC}}$ .....(i)	
$m\overline{AD} = m\overline{DB}$	Given
$\dfrac{m\overline{DB}}{m\overline{DB}} = \dfrac{m\overline{AE}}{m\overline{EC}}$	Put $m\overline{AD} = m\overline{DB}$ in (i)
$1 = \dfrac{m\overline{AE}}{m\overline{EC}}$	
$m\overline{AE} = m\overline{EC}$	

5. Prove that the line segment joining the mid-points of any two sides of a triangle is parallel to the third side.



Given:

In ΔABC, points D, E are such that $m\overline{AD} = m\overline{DB}$

$m\overline{AE} = m\overline{EC}$

$\dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{m\overline{AE}}{m\overline{EC}}$

To Prove:

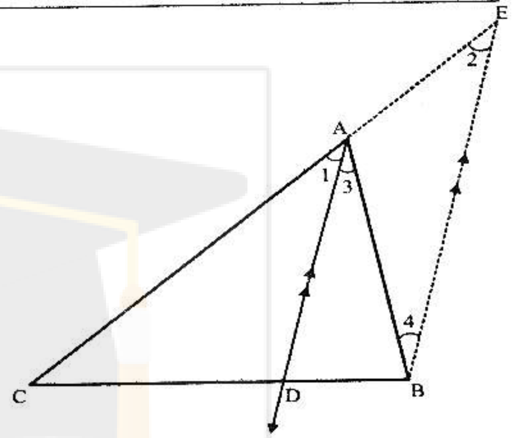
$\overline{DE} \parallel \overline{BC}$

Statements	Reasons
If $\overline{DE} \nparallel \overline{BC}$	
Then suppose $\overline{DE} \parallel \overline{BK}$	
Now $\dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{m\overline{AE}}{m\overline{EK}}$ .....(i)	
$\dfrac{m\overline{AD}}{m\overline{DB}} = \dfrac{m\overline{AE}}{m\overline{EC}}$ ......(ii)	Given
$\dfrac{m\overline{AE}}{m\overline{EK}} = \dfrac{m\overline{AE}}{m\overline{EC}}$	From (i) and (ii)
$m\overline{EK} = m\overline{EC}$	

It is possible only when point K lies on the point C.   Thus $\overline{DE} \parallel \overline{BC}$

Theorem

The internal bisector of an angle of a triangle divides the side opposite to it in the ratio of the lengths of the sides containing the angle.



Given: In ΔABC internal angle bisector of $\angle A$ meets $\overline{CB}$ at the point D.

To Prove: $m\overline{BD} : m\overline{DC} = m\overline{AB} : m\overline{AC}$

Construction:

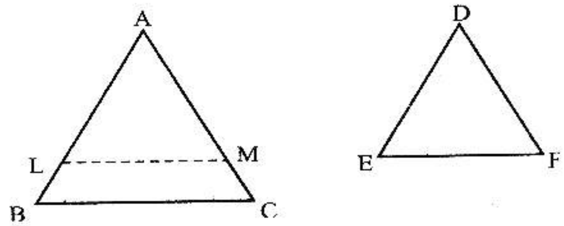
Draw a line segment $\overline{BE} \parallel \overline{DA}$ to meet $\overline{CA}$ produced at E.

Proof:

Statements	Reasons
$\because$ $\overline{AD} \parallel \overline{EB}$ and $\overline{EC}$ intersects them,	Construction
$\therefore$ $m\angle 1 = m\angle 2$ ....(i)	Corresponding angles
Again $\overline{AD} \parallel \overline{EB}$ and $\overline{AB}$ intersects them,	
$\therefore$ $m\angle 3 = m\angle 4$ ....(ii)	Alternate angles
But $m\angle 1 = m\angle 3$	Given
$\therefore$ $m\angle 2 = m\angle 4$	From (i) and (ii)
and $\overline{AB} \cong \overline{AE}$ or $\overline{AE} \cong \overline{AB}$	In a Δ, the sides opposite to congruent angles are also congruent.
Now $\overline{AD} \parallel \overline{EB}$	Construction
$\therefore$ $\dfrac{m\overline{BD}}{m\overline{DC}} = \dfrac{m\overline{EA}}{m\overline{AC}}$	By Theorem
or $\dfrac{m\overline{BD}}{m\overline{DC}} = \dfrac{m\overline{AB}}{m\overline{AC}}$	$m\overline{EA} = m\overline{AB}$ (proved)
Thus $m\overline{BD} : m\overline{DC} = m\overline{AB} : \overline{AC}$	

Theorem: If two triangles are similar, then the measures of their corresponding sides are proportional.

Given: $\Delta ABC \sim \Delta DEF$



i.e., $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$

To Prove:

$$\frac{m\overline{AB}}{m\overline{DE}} = \frac{m\overline{AC}}{m\overline{DF}} = \frac{m\overline{BC}}{m\overline{EF}}$$

Construction:

i) Suppose that $m\overline{AB} > m\overline{DE}$

ii) $m\overline{AB} \leq m\overline{DE}$

On $\overline{AB}$ take a point L such that $m\overline{AL} = m\overline{DE}$

On $\overline{AC}$ take a point M such that $m\overline{AM} = m\overline{DF}$. Join L and M by the line segment LM.

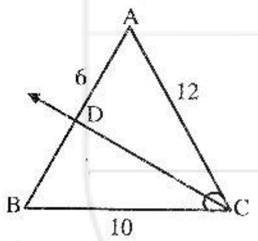
Proof:

Statements	Reasons
i) In $\Delta ALM \longleftrightarrow \Delta DEF$	
$\angle A \cong \angle D$	Given
$\overline{AL} \cong \overline{DE}$	Construction
$\overline{AM} \cong \overline{DF}$	Construction
Thus $\Delta ALM \cong \Delta DEF$	S.A.S. Postulate
and $\angle L \cong \angle E$, $\angle M \cong \angle F$,	(Corresponding angles of congruent triangles)
Now $\angle E \cong \angle B$, and $\angle F \cong \angle C$	Given
$\therefore$ $\angle L \cong \angle B$, $\angle M \cong \angle C$,	Transitivity of congruence
Thus $\overline{LM} \parallel \overline{BC}$	Corresponding angles are equal.
Hence $\dfrac{m\overline{AL}}{m\overline{AB}} = \dfrac{m\overline{AM}}{m\overline{AC}}$	By Theorem
or $\dfrac{m\overline{DE}}{m\overline{AB}} = \dfrac{m\overline{DF}}{m\overline{AC}}$ .....(i)	$m\overline{AL} = m\overline{DE}$ and $m\overline{AM} = m\overline{DF}$ (construction)
Similarly by intercepting segments on $\overline{BA}$ and $\overline{BC}$, we can prove that $\dfrac{m\overline{DE}}{m\overline{AB}} = \dfrac{m\overline{EF}}{m\overline{BC}}$ .....(ii)	
Thus $\dfrac{m\overline{DE}}{m\overline{AB}} = \dfrac{m\overline{DF}}{m\overline{AC}} = \dfrac{m\overline{EF}}{m\overline{BC}}$	by (i) and (ii)
or $\dfrac{m\overline{AB}}{m\overline{DE}} = \dfrac{m\overline{AC}}{m\overline{DF}} = \dfrac{m\overline{BC}}{m\overline{EF}}$	by taking reciprocals
ii) If $m\overline{AB} < m\overline{DE}$, it can similarly be	

proved by taking intercepts on the sides of ΔDEF	
If $m\overline{AB}=m\overline{DE}$,	
then in $\Delta ABC \longleftrightarrow \Delta DEF$	
$\angle A \cong \angle D$	Given
$\angle B \cong \angle E$	Given
and $\overline{AB} \cong \overline{DE}$	
so $\Delta ABC \cong \Delta DEF$	$A.S.A \cong A.S.A$
Thus $\dfrac{m\overline{AB}}{m\overline{DE}}=\dfrac{m\overline{AC}}{m\overline{DF}}=\dfrac{m\overline{BC}}{m\overline{EF}}=1$	$\overline{AC} \cong \overline{DF}$, $\overline{BC} \cong \overline{EF}$
Hence the result is true for all the cases.	

Exercise 14.2

1. In ΔABC as shown in the figure, $\overrightarrow{CD}$ bisects $\angle C$ and meets $\overline{AB}$ at D, $m\overline{BD}$ is equal to a) 5 b) 16 c) 10 d) 18

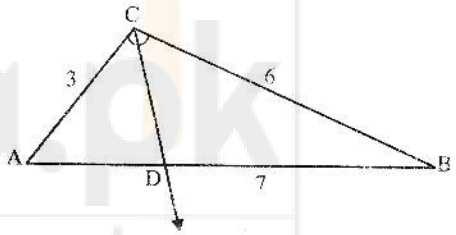


Ans. $\dfrac{m\overline{BD}}{m\overline{DA}}=\dfrac{m\overline{BC}}{m\overline{CA}}$

$\dfrac{m\overline{BD}}{6}=\dfrac{10}{12}$

$m\overline{BD}=\dfrac{10}{12}\times 6=5$

2. In ΔABC as shown in the figure, $\overrightarrow{CD}$ bisects $\angle C$. If $m\overline{AC}=3$, $m\overline{CB}=6$ and $m\overline{AB}=7$, then find $m\overline{AD}$ and $m\overline{DB}$.



Ans. $m\overline{AD}=x$

$m\overline{BD}=7-x$

$\dfrac{m\overline{AD}}{m\overline{DB}}=\dfrac{m\overline{AC}}{m\overline{CB}}$

$\dfrac{x}{7-x}=\dfrac{3}{6}$

$\dfrac{x}{7-x}=\dfrac{1}{2}$

$2x=1(7-x)$

$2x=7-x$

$3x=7 \Rightarrow \quad x=\dfrac{7}{3}$

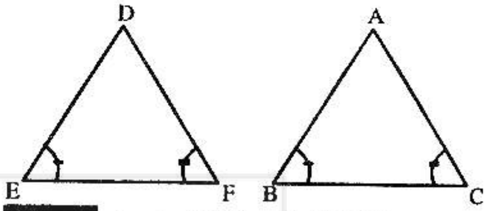
$m\overline{AD}=\dfrac{7}{3}$

$m\overline{DB}=7-x$

$$= 7 - \frac{7}{3}$$

$$= \frac{21-7}{3} = \frac{14}{3}$$

3. Show that in any correspondence of two triangles if two angles of one triangle are congruent to the corresponding angles of the other, then the triangles are similar.



Given: In ΔABC and ΔDEF

$m\angle B = m\angle E$

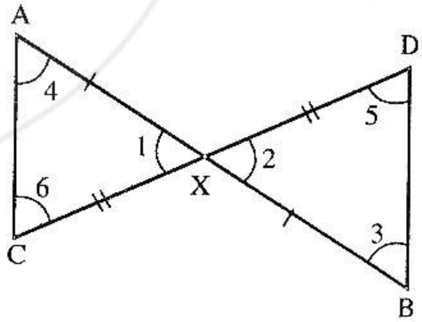
$m\angle C = m\angle F$

To Prove: $\Delta ABC \sim \Delta DEF$

Proof:

Statements	Reasons
$m\angle B + m\angle C + m\angle A = 180^o$ ---(i)	Sum of interior angles of triangle is 180^o
$m\angle E + m\angle F + m\angle D = 180^o$....(ii)	Given
$m\angle B + m\angle C + m\angle D = 180^o$...(iii) $m\angle A - m\angle D = 0$ $m\angle A = m\angle D$ All Angles of ΔDEF and ΔABC are congruent Thus $\Delta ABC \sim \Delta DEF$.	Subtracting (i) from (ii)

4. If line segments $\overline{AB}$ and are $\overline{CD}$ intersecting at point X and $\frac{m\overline{AX}}{m\overline{XB}} = \frac{m\overline{CX}}{m\overline{XD}}$ then show that ΔAXC and ΔBXD are similar.



Given:

$\overline{AB}$ and $\overline{CD}$ intersect each other at point x and

$$\frac{m\overline{AX}}{m\overline{XB}} = \frac{m\overline{CX}}{m\overline{XD}}$$

To Prove:

$\Delta AXC \sim \Delta BXD$

Proof:

Statements	Reasons
In ΔAXC and ΔBXD	
$\angle 1 \cong \angle 2$	Vertical angles
$\dfrac{m\overline{AX}}{m\overline{XB}} = \dfrac{m\overline{CX}}{m\overline{XD}}$	Given
Then $\overline{AC} \parallel \overline{BD}$	
$\angle 4 \cong \angle 3$	Alternate angles
$\angle 6 \cong \angle 5$	
Thus $\dfrac{m\overline{AX}}{m\overline{XB}} = \dfrac{m\overline{CX}}{m\overline{XD}} = \dfrac{m\overline{AC}}{m\overline{DB}}$	
Hence ΔAXC and ΔBXD are similar.	

5. Which of the following are true and which are false?

i. Congruent triangles are of same size and shape. True

ii. Similar triangles are of same shape but different sizes. True

iii. Symbol used for congruent is '~'. False

iv. Symbol used for similarity is '$\cong$'. False

v. Congruent triangles are similar. True

vi. Similar triangles are congruent. False

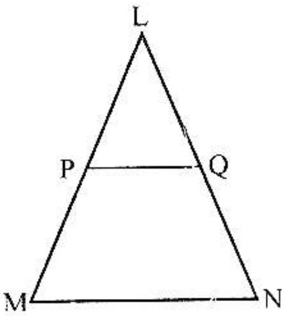
vii. A line segment has only one mid point. True

viii. One and only one line can be drawn through two points. True

ix. Proportion is non-equality of two ratios. False

x. Ratio has no unit. True

6. In ΔLMN show in the figure, $\overline{MN} \parallel \overline{PQ}$.



i) If $m\overline{LM} = 5\text{cm}$, $m\overline{LP} = 2.5\text{cm}$, $m\overline{LQ} = 2.3\text{cm}$, then find $m\overline{LN}$.

ii) If $m\overline{LM} = 6\text{cm}$, $m\overline{LQ} = 2.5cm$, $m\overline{QN} = 5\text{cm}$, then find $m\overline{LP}$.

Given: In ΔLMN, $\overline{MN} \parallel \overline{PQ}$

$m\overline{LM} = 5\text{cm}, m\overline{LP} = 2.5\text{cm}, m\overline{LQ} = 2.3\text{cm}$

To Prove: mLN = ?

Proof:

Statements	Reasons
$\dfrac{m\overline{LN}}{m\overline{LQ}} = \dfrac{m\overline{LM}}{m\overline{LP}}$	$\overline{PQ} \parallel \overline{MN}$ (Given)

$\frac{m\overline{LN}}{2.3}=\frac{5}{2.5}$ $m\overline{LN}=\frac{5\times 2.3}{2.5}$ $=\frac{5\times 23}{25}$ $=4.6\text{cm}$	Putting Values

(ii)

Given: $\Delta LMN, \overline{MN} \parallel \overline{PQ}$

$m\overline{QN}=5\text{cm}, m\overline{LQ}=2.5\text{cm}, m\overline{LM}=6\text{cm}.$

To prove: $m\overline{LP}=?$

Proof:

$$\frac{m\overline{LP}}{m\overline{LM}}=\frac{m\overline{LQ}}{m\overline{LN}}$$

$$\frac{m\overline{LP}}{m\overline{LM}}=\frac{m\overline{LQ}}{m\overline{LQ}+m\overline{QN}}$$

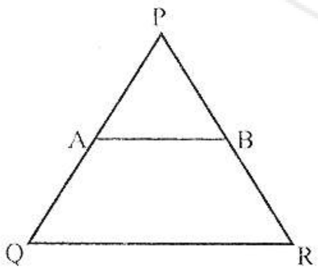
$$\frac{m\overline{LP}}{6}=\frac{2.5}{2.5+5}$$

$$m\overline{LP}=\frac{2.5}{7.5}\times 6$$

$$m\overline{LP}=\frac{1}{3}\times 6$$

$$=2\text{cm}.$$

7. In the shown figure, let $m\overline{PA}=8x-7$, $m\overline{PB}=4x-3$, $m\overline{AQ}=5x-3$, $m\overline{BR}=3x-1$. Find the value of x if $\overline{AB} \parallel \overline{QR}$.



If $\overline{AB} \parallel \overline{QR}$ then

$$\frac{m\overline{PA}}{m\overline{AQ}}=\frac{m\overline{PB}}{m\overline{BR}}$$

Putting values

$$\frac{8x-7}{5x-3}=\frac{4x-3}{3x-1}$$

$$(8x-7)(3x-1)=(5x-3)(4x-3)$$

$$24x^2-8x-21x+7=20x^2-15x-12x+9$$

$$24x^2-29x+7=20x^2-27x+9$$

$$24x^2-20x^2-29x+27x+7-9=0$$

$$4x^2-2x-2=0$$

$$2x^2-x-1=0$$

$$2x^2-2x+x-1=0$$

$$2x(x-1)+1(x-1)=0$$

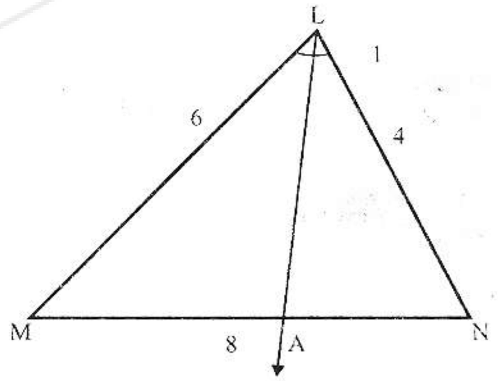
$$(2x+1)(x-1)=0$$

$$2x+1=0 \quad \text{or} \quad x-1=0$$

$$2x=-1 \qquad x=1$$

$$x=\frac{-1}{2}$$

8. In ΔLMN shown in the figure $\overline{LA}$ bisects $\angle L$. If $m\overline{LN}=4$, $m\overline{LM}=6$, $m\overline{MN}=8$, then find $m\overline{MA}$ and $m\overline{AN}$.



Given: In $\Delta LMN, \overline{LA}$ is angle bisector of $\angle L$.

$m\overline{LM} = 6cm, m\overline{LN} = 4cm, m\overline{MN} = 8cm.$

To Prove: $m\overline{MA} = ?$, $m\overline{AN} = ?$

Proof:

Let $m\overline{AN} = xcm$

$m\overline{MA} = 8 - xcm$

$\frac{m\overline{MA}}{m\overline{AN}} = \frac{m\overline{LM}}{m\overline{LN}}$

Putting values

$\frac{8-x}{x} = \frac{6}{4}$

$4(8-x) = 6x$

$32 - 4x = 6x$

$32 = 6x + 4x$

$10x = 32$

$x = \frac{32}{10} = 3.2$

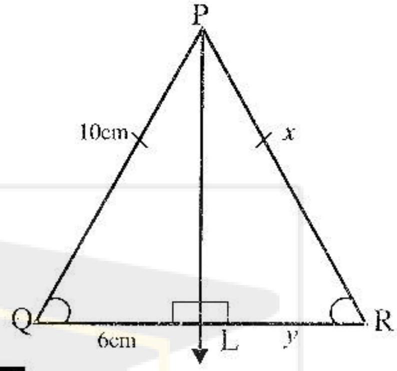
$\therefore \quad m\overline{AN} = 3.2cm.$

$m\overline{MA} = 8 - x$

$= 8 - 3.2$

$= 4.8cm.$

9. In Isosceles ΔPQR shown in the figure, find the value of x and y.



Given:

In $\Delta PQR, \overline{PQ} \cong \overline{PR}$ and $\overline{PL} \perp \overline{QR}$.

To Prove: $x = ?$ $y = ?$

Proof:

In ΔPRL and ΔPQL

$m\overline{PQ} = m\overline{PR}$ ... (i)	Isosceles triangle
$m\angle PLQ = m\angle PLR$	Each of right angle
$m\overline{PL} = m\overline{PL}$	Common
$\Delta PQL \cong \Delta PRL$	H.S. ≅ H.S

$m\overline{QL} = m\overline{LR}$

$6 = y$

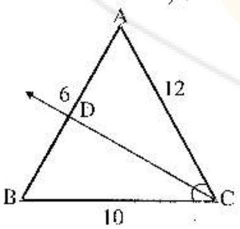
$\Rightarrow y = 6cm.$

From (i) $x = 10cm.$

OBJECTIVE

1. In ΔABC as shown in figure, $\overrightarrow{CD}$ bisects $\angle C$ and meets $\overline{AB}$ at D, a $m\overline{BD}$ is equal to:

(a) 5
(b) 16
(c) 10
(d) 18



2. In ΔABC shown in figure, $\overrightarrow{CD}$ bisects $\angle C$, if $m\overline{AC} = 3$, $m\overline{CB} = 6$ and $m\overline{AB} = 7$ then

(i) $\overline{AD}$ = ___

(a) $\frac{7}{3}$ (b) $\frac{14}{3}$

(c) $\frac{9}{2}$ (d) $\frac{11}{2}$

(ii) $m\overline{BD}$ = ___

(a) $\frac{7}{3}$ (b) $\frac{14}{3}$

(c) $\frac{15}{2}$ (d) $\frac{11}{2}$

3. One and only one line can be drawn through ___ points:

(a) Two (b) Three

(c) Four (d) Five

4. The ratio between two alike quantities is defined as:

(a) a : b

(b) b : a

(c) a : b = c : d

(d) None

5. If a line segment intersects the two sides of a triangle in the same ratio then it is parallel to the __ side:

(a) Third (b) Fourth

(c) Second (d) None

6. Two triangles are said to be similar if these are equiangular and their corresponding sides are ___

(a) Proportional

(b) congruent

(c) concurrent

(d) None

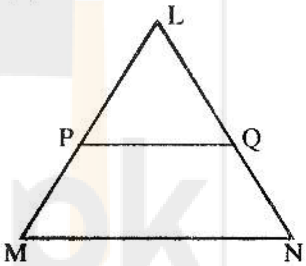
7. In ΔLMN shown in the figure $\overline{MN} \parallel \overline{PQ}$ if $m\overline{LM} = 5cm$, $m\overline{LP} = 2.5cm$, $m\overline{LQ} = 2.3cm$ then $m\overline{LN} =$ ___ :

(a) 4.6cm

(b) 4.5cm

(c) 3.5cm

(d) 4.0



ANSWER KEY

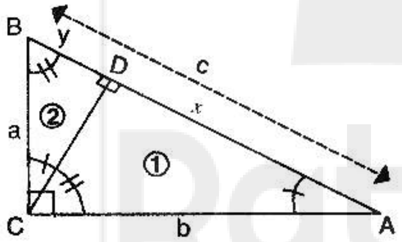
1.	a	**2.**	(i) a (ii) b	**3.**	a	**4.**	a	**5.**	a
6.	a	**7.**	a						

Unit 15

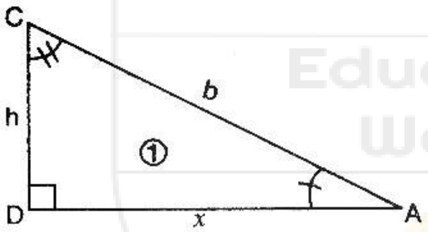
PYTHAGORAS THEOREM

Pythagoras Theorem

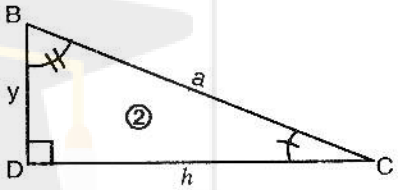
In a right angled triangle, the square of the length of hypotenuse is equal to the sum of the squares of the lengths of the other two sides.



(i)



(ii) a



(ii)-b

Given

ΔACB is a right angled triangle in which $m\angle C = 90^\circ$ and $m\overline{BC} = a$, $m\overline{AC} = b$ and $m\overline{AB} = c$.

To Prove

$$c^2 = a^2 + b^2$$

Construction

Draw $\overline{CD}$ perpendicular from C on $\overline{AB}$.

Let $m\overline{CD} = h$, $m\overline{AD} = x$ and $m\overline{BD} = y$.

Line segment CD splits ΔABC into two Δs ADC and BDC which are separately shown in the figures (ii)-a and (ii)-b respectively.

Proof (Using similar Δs)

Statements	Reasons
In $\Delta ADC \longleftrightarrow \Delta ACB$	Refer to figure(ii)-a and (i)
$\angle A \cong \angle A$	Common – self congruent
$\angle ADC \cong \angle ACB$	Construction – given, each angle = 90°
$\angle C \cong \angle B$	$\angle C$ and $\angle B$, complements of $\angle A$.
$\therefore$ $\Delta ADC \sim \Delta ACB$	Congruency of three angles
$\therefore$ $\dfrac{x}{b} = \dfrac{b}{c}$	(Measures of corresponding sides of similar triangles are proportional)
or $x = \dfrac{b^2}{c}$ ....(i)	

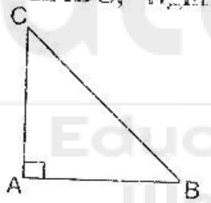
Again in $\Delta BDC \longleftrightarrow \Delta BCA$	Refer to figure (ii)-b and (i)
$\angle B \cong \angle B$	Common–self congruent
$\angle BDC \cong \angle BCA$	Construction –given, each angle = 90°
$\angle C \cong \angle A$	$\angle C$ and $\angle A$, complements of $\angle B$
$\therefore \quad \Delta BDC \sim \Delta BCA$	Congruency of three angles.
$\therefore \quad \frac{y}{a} = \frac{a}{c}$	(Corresponding sides of similar triangles are proportional).
or $\quad y = \frac{a^2}{c}$ ....(ii)	
But $\quad y + x = c$	Supposition.
$\therefore \quad \frac{a^2}{c} + \frac{b^2}{c} = c$	By (i) and (ii)
or $\quad a^2 + b^2 = c^2$	Multiplying both sides by c.
i.e., $\quad c^2 = a^2 + b^2$	

Corollary

In a right angled ΔABC, right angled at A.

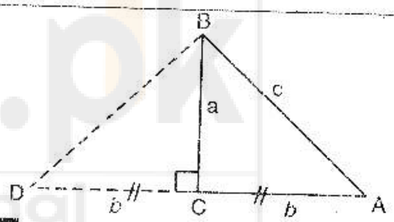
(i) $\overline{AB}^2 = \overline{BC}^2 - \overline{CA}^2$

(ii) $\overline{AC}^2 = \overline{BC}^2 - \overline{AB}^2$



Converse of Pythagoras' Theorem

If the square of one side of a triangle is equal to the sum of the squares of the other two sides then the triangle is a right angled triangle.



Given In a ΔABC, $m\overline{AB} = c, m\overline{BC} = a$ and $m\overline{AC} = b$ such that $a^2 + b^2 = c^2$.

To Prove ΔACB is a right angled triangle.

Construction Draw $\overline{CD}$ perpendicular to $\overline{BC}$ such that $\overline{CD} \cong \overline{CA}$. Join the points B and D.

Proof

Statements	Reasons
ΔDCB is a right –angled triangle.	Construction
$\therefore \quad (m\overline{BD})^2 = a^2 + b^2$	Pythagoras theorem
But $\quad a^2 + b^2 = c^2$	Given
$\therefore \quad (m\overline{BD})^2 = c^2$	Taking square root of both sides.
or $\quad m\overline{BD} = c$	
Now in $\Delta DCB \leftrightarrow \Delta ACB$	
$\overline{CD} \cong \overline{CA}$	Construction

$\overline{BC} \cong \overline{BC}$	Common
$\overline{DB} \cong \overline{AB}$	Each side = c.
$\therefore$ $\Delta DCB \cong \Delta ACB$	S.S.S. $\cong$ S.S.S.
$\therefore$ $\angle DCB \cong \angle ACB$	(Corresponding angles of congruent triangles)
But $m\angle DCB = 90^\circ$	Construction
$\therefore$ $m\angle ACB = 90^\circ$	
Hence the ΔACB is a right-angled triangle.	

Corollary: Let c be the longest of the sides a, b and c of a triangle.

- If $a^2 + b^2 = c^2$, then the triangle is right.
- If $a^2 + b^2 > c^2$, then the triangle is acute.
- If $a^2 + b^2 < c^2$, then the triangle is obtuse.

Exercise 15

1. Verify that the Δs having the following measures of sides are right-angled.

(i) $a = 5$ cm, $b = 12$ cm, $c = 13$ cm

Ans. $(Hyp)^2 = (Perp.)^2 + (Base)^2$

$(13)^2 = (12)^2 + (5)^2$

$169 = 144 + 25$

$169 = 169$

$\therefore$ The triangle is right angled.

(ii) $a = 1.5$ cm, $b = 2$ cm, $c = 2.5$ cm

Ans. $(Hyp)^2 = (Perp.)^2 + (Base)^2$

$(2.5)^2 = (1.5)^2 + (2)^2$

$625 = 2.25 + 4$

$6.25 = 6.25$

$\therefore$ The triangle is right angled.

(iii) $a = 9$ cm, $b = 12$ cm, $c = 15$ cm

Ans. $(Hyp)^2 = (Perp.)^2 + (Base)^2$

$(15)^2 = (12)^2 + (9)^2$

$225 = 144 + 81$

$225 = 225$

$\therefore$ The triangle is right angled.

(iv) $a = 16$ cm, $b = 30$ cm, $c = 34$ cm

Ans. $(Hyp)^2 = (Perp.)^2 + (Base)^2$

$(34)^2 = (30)^2 + (16)^2$

$1156 = 900 + 256$

$1156 = 1156$

$\therefore$ The triangle is right angled.

2. Verify that $a^2 + b^2$, $a^2 - b^2$ and $2ab$ are the measures of the sides of a right angled triangle where a and b are any two real numbers ($a > b$).

Ans. In right angle triangle.

$H^2 = P^2 + B^2$

$(a^2 + b^2)^2 = a^4 + b^4 + 2a^2b^2$ ....(i)

$(a^2 - b^2)^2 = a^4 + b^4 - 2a^2b^2$ ....(ii)

$(2ab)^2 = 4a^2b^2$ .... (iii)

Adding (ii) and (iii) we get

$(a^2 - b^2)^2 + (2ab)^2 = a^4 + b^4 - 2a^2b^2 + 4a^2b^2$

$= a^4 + b^4 + 2a^2b^2$ ....(iv)

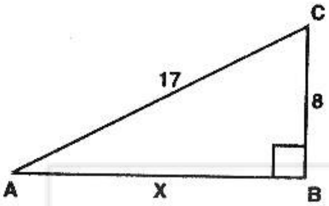
Comparing (i) and (iv), we get

$(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2$

Hence $a^2 + b^2$, $a^2 - b^2$ and $2ab$ are measures of the sides of a right angled triangle where $a^2 + b^2$ is Hypotenuse.

3. The three sides of a triangle are of measure 8, x and 17 respectively. For what value of x will it become base of a right angled triangle?

Ans:



Consider a right angled triangle

With $\overline{AB} = x$

$\overline{BC} = 8$

and $\overline{AC} = 17$

If x is the base of right angled ΔABC then we know by Pythagoras theorem that

$$(hyp)^2 = (Base)^2 + (perp)^2$$
$$(17)^2 = x^2 + (8)^2$$
$$289 = x^2 + 64$$
$$x^2 + 64 = 289$$
$$x^2 = 289 - 64$$
$$x^2 = 225$$
$$x = \sqrt{225}$$

As x is measure of side

So $x = 15$ units

4. In an isosceles Δ, the base $\overline{BC} = 28$ cm, and $\overline{AB} = \overline{AC} = 50$cm.

If $\overline{AD} \perp \overline{BC}$, then find:

(i) Length of $\overline{AD}$

(ii) Area of ΔABC

Given

$m\overline{AC} = m\overline{AB} = 50\,cm$

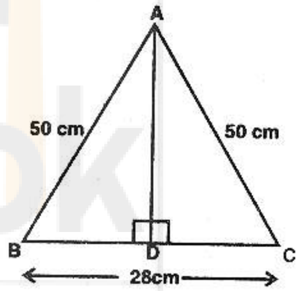
$m\overline{BC} = 28cm$

$\overline{AD} \perp \overline{BC}$

To Prove

$m\overline{AD} = ?$

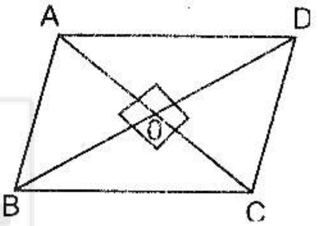
Area of $\Delta ABC = ?$



Proof

Statements	Reasons
In right angled triangle	
$m\overline{CD} = 14cm$	$\overline{CD} = \frac{1}{2} m\overline{BC}$
$m\overline{AC} = 50cm$	Given
$(mAD)^2 = (mAC)^2 - (mCD)^2$	$(mAC)^2 = (mAD)^2 - (mCD)^2$ (by Pythagoras theorem)
$(mAD)^2 = (50)^2 - (14)^2$	
$= 2500 - 196$	
$= 2304$	
$\sqrt{(mAD)^2} = \sqrt{2304}$	Taking square root of both sides
$m\overline{AD} = 18\,cm$	
(ii) Area of $\Delta ABC = \frac{Base \times Altitude}{2}$	
$= \frac{28 \times 48}{2}$	
$= 14 \times 28$	
$= 672$ sq.cm	

In a quadrilateral ABCD, the diagonals $\overline{AC}$ and $\overline{BD}$ are perpendicular to each other. Prove that:



$$m\overline{AB}^2 + m\overline{CD}^2 = m\overline{AD}^2 + m\overline{BC}^2.$$

Given: Quadrilateral ABCD diagonal $\overline{AC}$ and $\overline{BD}$ are perpendicular to each other.

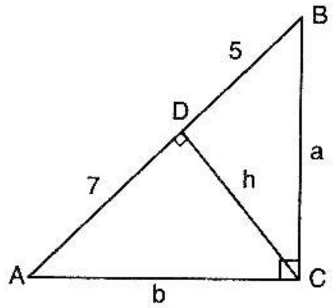
To Prove:

$$\mathbf{m(\overline{AB})^2 + m(\overline{CD})^2 = m(\overline{AD})^2 + m(\overline{BC})^2}$$

Proof

Statements	Reasons
In right triangle AOB $m(\overline{AB})^2 = m(\overline{AO})^2 + m(\overline{OB})^2$ ....,(i)	By Pythagoras theorem
In right triangle COD $m(\overline{CD})^2 = m(\overline{OC})^2 + m(\overline{OD})^2$ ....,(ii)	By Pythagoras theorem
In right triangle AOD $m(\overline{AD})^2 = m(\overline{AO})^2 + m(\overline{OD})^2$ ....,(iii)	By Pythagoras theorem
In right triangle BOC $m(\overline{BC})^2 = m(\overline{OB})^2 + m(\overline{OC})^2$ ....,(iv)	By Pythagoras theorem
$m(\overline{AB})^2 + m(\overline{CD})^2 = m(\overline{AO})^2 + m(\overline{OB})^2 + m(\overline{OC})^2 + m(\overline{OD})^2$ ....,(v)	By adding (i) and (ii)
$m(\overline{AD})^2 + m(\overline{BC})^2 = m(\overline{AO})^2 + m(\overline{OD})^2 + m(\overline{OB})^2 + m(\overline{OC})^2$ ....(vi)	By adding (ii) and (iv)
$(m\overline{AB})^2 + (m\overline{CD})^2 = (m\overline{BC})^2 + (m\overline{AD})^2$	By adding (v) and (vi)

6. (i) In the ΔABC as shown in the figure, $m\angle ACB = 90^o$ and $\overline{CD} \perp \overline{AB}$. Find the lengths a, h and b if $m\overline{BD} = 5$ units and $m\overline{AD} = 7$ units.



Given: A ΔABC as shown

$m\ \angle ACB = 90^0$

and $\overline{CD} \perp \overline{AB}$

To prove : a, h and b.

In right angled ΔBDC

$a^2 = 25 + h^2$ .... (i)

in right angled ΔADC

$b^2 = 49 + h^2$ .... (ii)

in right angled ΔABC

$a^2+b^2 = 144$ .... (iii)

adding (i) and (ii)

$a^2+b^2 = 74+2h^2$ .... (iv)

from (iii) and (iv)

$$74 + 2h^2 = 144$$
$$2h^2 = 144-74$$
$$2h^2 = 70$$
$$h^2 = 35$$
$$h = \sqrt{35}$$

Now we will find a and b

Put $\quad h^2 = 35$ (in Eq. 1)

$$a^2 = 25+35$$
$$a^2 = 60$$
$$a = \sqrt{60}$$
$$= \sqrt{4\times 15}$$
$$a = 2\sqrt{15}$$

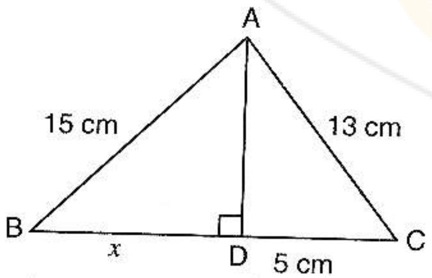
now put $\quad h^2 = 35$ (in Eq. 2)

$$b^2 = 49+35$$
$$b^2 = 48$$
$$b = \sqrt{84}$$
$$b = \sqrt{4\times 21}$$
$$b = 2\sqrt{21}$$

SO $\quad a = 2\sqrt{15}$

$$h = \sqrt{35}$$
$$b = 2\sqrt{21}$$

(ii) Find the value of x in the shown in the figure.



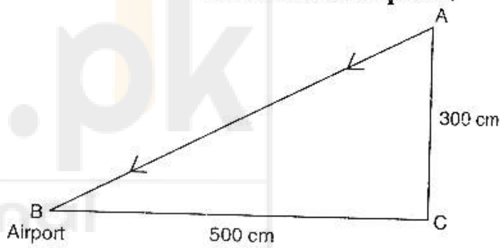
In right angled triangle ADC

$$m(\overline{AC})^2 = m(\overline{AD})^2 + m(\overline{DC})^2$$
$$(13)^2 = (AD)^2 + (5)^2$$
$$169 = (AD)^2 + 25$$
$$(AD)^2 = 169 - 25$$
$$(AD)^2 = 144$$
$$AD = \sqrt{144}$$
$$AD = 12cm$$

In right angled triangle ABD

$$(AB)^2 = (AD)^2 + (BD)^2$$
$$(15)^2 = (12)^2 + x^2$$
$$225 = 144 + x^2$$
$$x^2 = 225 - 144$$
$$x^2 = 81$$
$$x = 9 \text{ cm}$$

7. A plane is at a height of 300 m and is 500 m away from the airport as shown in the figure. How much distance will it travel to land at the airport?



Here A be the position of plane and B be the position of airport.

$$m\overline{AC} = 500m$$
$$m\overline{BC} = 300m$$
$$m\overline{AB} = ?$$

Applying Pythagoras theorem on right angled triangle ABC

$$|\overline{AB}|^2 = |\overline{AC}|^2 + |\overline{BC}|^2$$
$$= (500)^2 + (300)^2$$
$$= 250000 + 90000$$
$$= 34000$$
$$|\overline{AB}|^2 = 34\times 10000$$

so $\quad |\overline{AB}| = \sqrt{34\times 10000}$

$$= \sqrt{34\times 100\times 100}$$

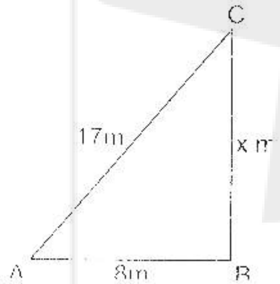
$= 100\sqrt{34}m$

So required distance is $100\sqrt{34}m$

8. A ladder 17 m long rests against a vertical wall. The foot of the ladder is 8m away from the base of the wall. How high up the wall will the ladder reach?

Ans. Let the height of ladder = x m

in right angled triangle



$(\text{Hyp})^2 = (\text{Perp.})^2 + (\text{Base})^2$

$(17)^2 = (x)^2 + (8)^2$

$289 = x^2 + 64$

$x^2 = 289 - 64$

$x^2 = 225$

$x = \sqrt{225} = 15\text{m}$

9. A student travels to his school by the route as shown in the figure. Find $m\overline{AD}$, the direct distance from his house to school.

According to figure, $m\overline{AB} = 2\text{km}$

$m\overline{BC} = 6\text{km}$

$m\overline{CD} = 3\text{km}$

Here $m\overline{AB}$ and $m\overline{CD}$ are perpendicular

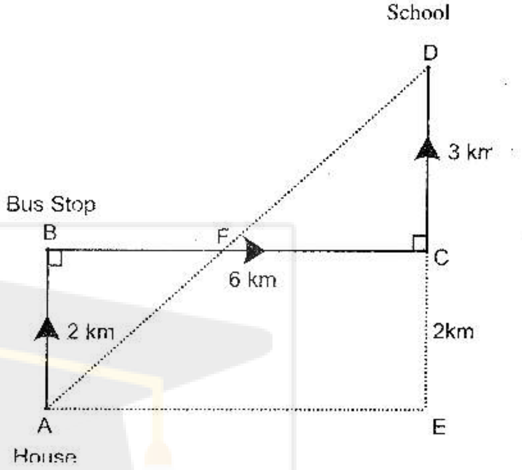
Perpendicular $= \overline{AB} + \overline{CD}$

$= 2 + 3$

$= 5\text{km}$

According to Pythagoras theorem

$(H)^2 = P^2 + B^2$



$(m\overline{AD})^2 = (5)^2 + (6)^2 = 25 + 36$

$(m\overline{AD})^2 = 61$

$m\overline{AD} = \sqrt{61}\text{ Km}$

10. Which of the following are true and which are false?

(i) In a right angled triangle greater angle is $90°$. (T)

(ii) In a right angled triangle right angle is $60°$. (F)

(iii) In a right triangle hypotenuse is a side opposite to right angle. (T)

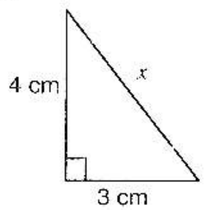
(iv) If a, b, c are sides of right angled triangle with c as longer side then $c^2 = a^2 + b^2$. (T)

(v) If 3 cm and 4 cm are two sides of a right angled triangle, then hypotenuse is 5 cm. (T)

(vi) If hypotenuse of an isosceles right triangle is $\sqrt{2}$ cm then each of other side is of length 2 cm. (F)

11. Find the unknown value in each of the following figures.

(i)



By Pythagoras theorem

$(Hyp)^2 = (Perp.)^2 + (Base)^2$

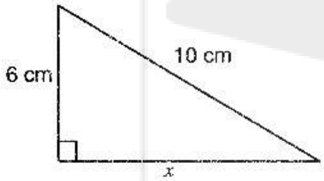
$x^2 = (4)^2 + (3)^2$

$x^2 = 16 + 9$

$x^2 = 25 \Rightarrow x = \sqrt{25}$

$x = 5cm$

(ii)



By Pythagoras theorem

$(Hyp)^2 = (Perp.)^2 + (Base)^2$

$(10)^2 = (6)^2 + (x)^2$

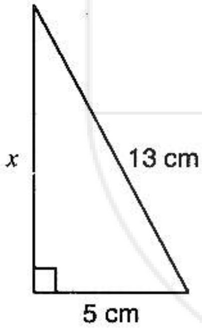
$100 = 36 + x^2$

$x^2 = 64$

$x = \sqrt{64}$

$X = 8cm$

(iii)



By Pythagoras theorem

$(Hyp)^2 = (Perp.)^2 + (Base)^2$

$(13)^2 = (x)^2 + (2)^2$

$169 = x^2 + 25$

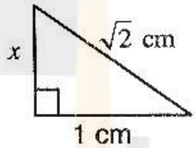
$x^2 = 169 - 25$

$x^2 = 144$

$x = \sqrt{144}$

$x = 12cm$

(iv)



By Pythagoras theorem

$(Hyp.)^2 = (Perp.)^2 + (Base)^2$

$(\sqrt{2})^2 = (x)^2 + (1)^2$

$(\sqrt{2})^2 = (x)^2 + (1)^2$

$2 = x^2 + 1$

$x^2 = 2 - 1$

$x^2 = 1$

$x = \sqrt{1} = 1cm$

OBJECTIVE

1. In a right angled triangle, the square of the length of hypotenuse is equal to the ____ of the squares of the lengths of the other two sides

(a) Sum
(b) Difference
(c) Zero
(d) None

2. If the square of one side of a triangle is equal to the sum of the squares of the other two sides then the triangle is a _____ triangle.

(a) Right angled
(b) Acute angled
(c) Obtuse angled
(d) None

3. Let c be the longest of the sides a, b and c of a triangle. If $a^2 + b^2 = c^2$, then the triangle is ____:
   (a) Right
   (b) Acute
   (c) Obtuse
   (d) None

4. Let c be the longest of the sides a, b and c of a triangle. If $a^2 + b^2 > c^2$ then triangle is:
   (a) Acute
   (b) Right
   (c) Obtuse
   (d) None

5. Let c be the longest of the sides a, b and c of a triangle of $a^2+b^2 < c^2$, then the triangle is:
   (a) Acute
   (b) Right
   (c) Obtuse
   (d) None

6. If 3cm and 4cm are two sides of a right angled triangle, then hypotenuse is;
   (a) 5cm
   (b) 3cm
   (c) 4cm
   (d) 2cm

7. In right triangle ____ is a side opposite to right angle.
   (a) Base
   (b) Perpendicular
   (c) Hypotenuse
   (d) None

ANSWER KEY

1.	a	**2.**	a	**3.**	a	**4.**	a	**5.**	c
6.	a	**7.**	c						

Unit 16 THEOREMS RELATED WITH AREA

Area of a Figure

The region enclosed by the bounding lines of a closed figure is called the area of the figure.

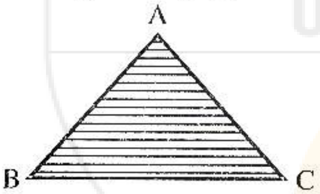
The area of a closed region is expressed in square units (say, sq. m or m^2) i.e., a positive real number.

Triangular region

The interior of a triangle is the part of the plane enclosed by the triangle.

A triangular region is the union of a triangle and its interior i.e., the three line segments forming the triangle and its interior.

By area of a triangle, we mean the area of its triangular region.



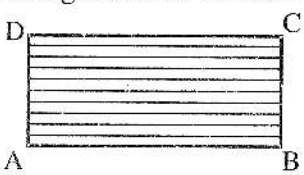
Congruent Area Axiom

If $\Delta ABC \cong \Delta PQR$, then area of (region ΔABC) = area of (region ΔPQR)

Define Rectangular Region

The interior of a rectangle is the part of the plane enclosed by the rectangle.

A rectangular region is the union of a rectangle and its interior.



A rectangular region can be divided into two or more than two triangular regions in many ways.

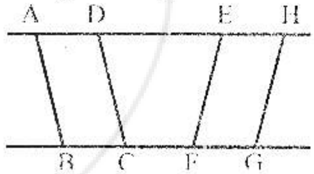
Note

If the length and width of a rectangle are a units and b units respectively, then the area of the rectangle is equal to $a \times b$ square units.

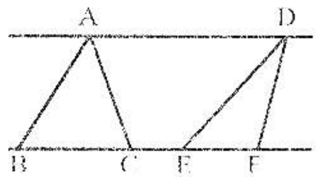
If a is the side of a square, its area $= a^2$, square units.

Between the same Parallels

(i) Two parallelograms are said to be between the same parallels, when their bases are in the same straight line and their sides opposite to these bases are also in a straight line; as the parallelograms ABCD, EFGH in the given figure.

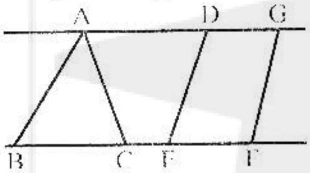


(ii) Two triangles are said to be between the same parallels, when their bases are in the same straight line and the line joining their vertices is parallel to their bases; as the Δ ABC, Δ DEF in the given figure.



(iii) A triangle and a parallelogram are said to be between the same parallels,

when their bases are in the same straight line, and the side of the parallelogram opposite the base, produced if necessary, passes through the vertex of the triangle as are the ΔABC and the parallelogram DEFG in the given figure.



Altitude of Parallelogram

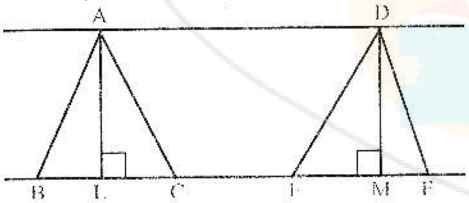
If one side of a parallelogram is taken as its base, the perpendicular distance between that side and the side parallel to it, is called the Altitude or Height of the parallelogram.

Altitude of the triangle

If one side of a triangle is taken as its base, the perpendicular to that side, from the opposite vertex is called the Altitude or Height of the triangle.

Example

"Triangles or parallelograms having the same or equal altitudes can be placed between the same parallels and conversely."



Place the triangles ABC, DEF so that their bases $\overline{BC}$, $\overline{EF}$ are in the same straight line and the vertices on the same side of it and suppose $\overline{AL}$, $\overline{DM}$ are the equal altitudes. We have to show that $\overline{AD}$ is parallel to BCEF.

Proof

$\overline{AL}$ and $\overline{DM}$ are parallel, for they are both perpendicular to $\overline{BF}$. Also $m\overline{AL} = m\overline{DM}$. (given)

$\therefore$ $\overline{AD}$ is parallel to $\overline{LM}$. A similar proof may be given in the case of parallelograms.

Note:

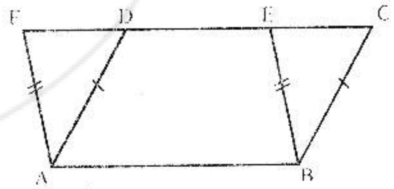
A diagonal of a parallelogram divides it into two congruent triangles (SSS) and hence of equal area.

Theorem

Parallelograms on the same base and between the same parallel lines (or of the same altitude) are equal in area.

Given

Two parallelograms ABCD and ABEF having the same base $\overline{AB}$ and $\overline{DE}$ between the same parallel lines $\overline{AB}$ and $\overline{DE}$.



To Prove

Area of parallelogram ABCD = area of parallelogram ABEF

Proof

Statements	Reasons
Area of (parallelogram ABCD) = Area of (quad. ABED) + area of (ΔCBE)..(i)	[Area addition axiom]

Area of (parallelogram ABEF)	
= area of (quad. ABED) + area of (ΔDAF)..(ii)	[Area addition axiom]
In Δs CBE and DAF	
$m\overline{CB} = m\overline{DA}$	[opposite sides of a parallelogram]
$m\overline{BE} = m\overline{AF}$	[opposite sides of a parallelogram]
$\angle CBE = \angle DAF$	[$\because \overline{BC} \parallel \overline{AD}, \overline{BE} \parallel \overline{AF}$]
$\therefore \Delta CBE \cong \Delta DAF$	[S.A.S. cong. Axiom]
$\therefore$ area of (ΔCBE) = area of (ΔDAF).........(iii)	[cong. Area axiom]
Hence area of (parallelogram ABCD) = area of (parallelogram ABEF)	From (i), (ii) and (iii)

Example

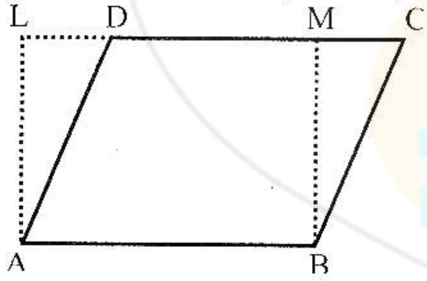
(i) The area of a parallelogram is equal to that of a rectangle on the same base and having the same altitude.

(ii) Hence area of parallelogram = base $\times$ altitude

Proof

Let ABCD be a parallelogram. $\overline{AL}$ is an altitude corresponding to side $\overline{AB}$.

(i) Since parallelogram ABCD and rectangle ALMB are on the same base $\overline{AB}$ and between the same parallels,



$\therefore$ by above theorem it follows that area of (parallelogram ABCD) = area of (rect. ALMB)

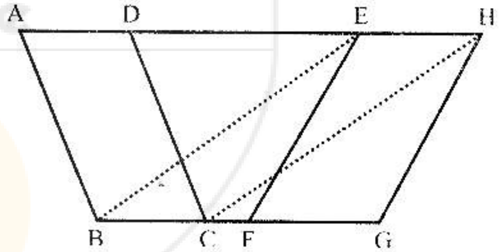
(ii)But area of (rect. ALMB) = $\overline{AB} \times \overline{AL}$

Hence

Area of (parallelogram ABCD) = $\overline{AB} \times \overline{AL}$

Theorem

Parallelograms on equal bases and having the same (or equal) altitude are equal in area.



Given

Parallelograms ABCD, EFGH are on the equal bases $\overline{BC}$, $\overline{FG}$, having equal altitudes.

To Prove

Area of (parallelogram ABCD) = area of (parallelogram EFGH)

Construction

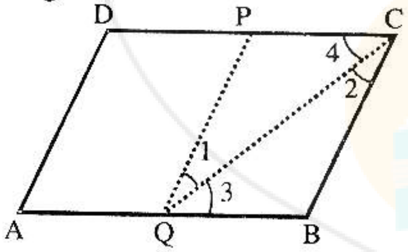
Place the parallelograms ABCD and EFGH so that their equal bases $\overline{BC}$, $\overline{FG}$ are in the straight line BCFG. Join $\overline{BE}$ and $\overline{CH}$.

Proof

Statements	Reasons
The given $\|\|^{gms}$ ABCD and EFGH are between the same parallels	Their altitudes are equal (given)
Hence ADEH is a straight line $\|\|$ to $\overline{BC}$	
$\therefore \quad m\overline{BC} = m\overline{FG}$	Given
$= m\overline{EH}$	EFGH is a parallelogram
Now $m\overline{BC} = m\overline{EH}$ and they are $\|\|$	
$\therefore \quad \overline{BE}$ and $\overline{CH}$ are both equal and $\|\|$ Hence EBCH is a parallelogram	A quadrilateral with two opposite sides congruent and parallel is a parallelogram
Now $\|\|^{gm}$ ABCD $= \|\|^{gm}$ EBCH ...(i)	Being on the same base $\overline{BC}$ and between the same parallels
But $\|\|^{gm}$ EBCH $= \|\|^{gm}$ EFGH ...(ii)	Being on the same base $\overline{EH}$ and between the same parallels
Hence area ($\|\|^{gm}$ ABCD) = area ($\|\|^{gm}$ EFGH)	From (i) and (ii)

Exercise 16.1

(1) Show that the line segment joining the mid-points of opposite sides of a parallelogram, divides it into two equal parallelograms.



Given ABCD is parallelogram. point p is midpoint of side $\overline{DC}$ i.e. $\overline{DP} \cong \overline{PC}$ and point Q is midpoint of side $\overline{AB}$ i.e. $\overline{AQ} \cong \overline{QB}$.

To Prove

Parallelogram AQPD $\cong$ parallelogram QBCP

Construction

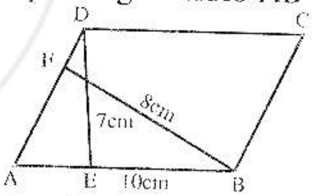
Join P to Q and Q to C.

Proof

Statements	Reasons
$m\overline{AB} = m\overline{DC}$	
$\frac{1}{2} m\overline{AB} = \frac{1}{2} m\overline{DC}$	Dividing by 2
$m\overline{QB} = m\overline{PC}$	

Now	
$\Delta PQC \leftrightarrow \Delta QBC$	
$\overline{QC} \cong \overline{QC}$	Common
$\overline{QB} \cong \overline{PC}$	Proved
$\angle 3 \cong \angle 4$	Alt. Angles $\overline{AB} \parallel \overline{DC}$
$\Delta PQC \cong \Delta QBC$	S.A.S = S.A.S
$\overline{PQ} \cong \overline{CB}$ ……………………(i)	Corresponding sides of congruent triangles
$\overline{AD} \cong \overline{CB}$ …………………(ii)	
$\overline{PQ} \cong \overline{AD} \cong \overline{CB}$	
$\angle 1 \cong \angle 2$	Corresponding angles of congruent triangles
$m\angle 1 + m\angle 3 = m\angle 2 + m\angle 4$	Corresponding angles of ‖gm
$\angle PQB \cong \angle PCB$	
$\angle A \cong \angle PCB$	
$\angle A \cong \angle PQB$	
Now	
‖gm AQPD and ‖gm QBCP	
$\overline{AQ} \cong \overline{QB}$	Given
$\overline{AD} \cong \overline{PQ}$	Proved
$\angle A \cong \angle PQB$	
Thus ‖gm AQPD $\cong$ ‖gm QBCP	

(2) In a parallelogram ABCD, $m\overline{AB} = 10$cm. The altitudes corresponding to sides AB and AD are respectively 7 cm and 8 cm. Find $\overline{AD}$.



Given Parallelogram ABCD, $m\overline{AB} = 10$cm altitudes. Corresponding to the sides $\overline{AB}$ and $\overline{AD}$ are 7cm and 8cm.

To Prove: $m\overline{AD} = ?$

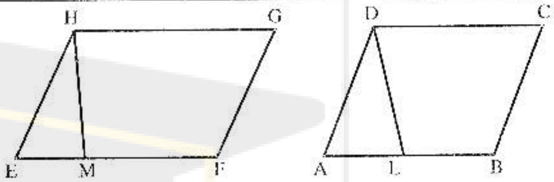
Construction Make ‖gm ABCD and show the given altitudes $\overline{DE} = 7$cm, $\overline{BF} = 8$cm.

Proof The area of parallelogram = base x altitude

Statements	Reasons
∴Area of parallelogram ABCD = 10 x 7 ………(i)	
Also area of the ‖gm ABCD = $\overline{AD}$ x 8 ……..(ii)	
∴ $m\overline{AD} \times 8 = 10 \times 7$	

$m\overline{AD} = \frac{10 \times 7}{8}$ $m\overline{AD} = \frac{35}{4} = 8\frac{3}{4}$ cm	

(3) If two parallelograms of equal areas have the same or equal bases, their altitudes are equal.



Given Two parallelograms of same or equal bases and same areas.

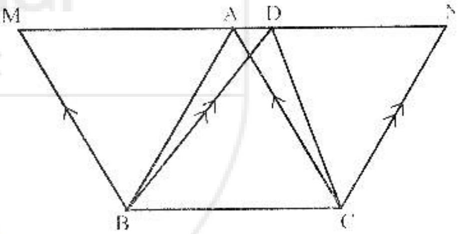
To Prove Their altitudes are equal.

Construction Make the ||gm ABCD and EFGH. Draw $\overline{DL} \perp \overline{AB}$ and $\overline{HM} \perp \overline{EF}$

Proof

Statements	Reasons
Area of the ||gm ABCD = area of the ||gm EFGH	
base x altitude = base x altitude	
$m\overline{AB} \times m\overline{DL} = m\overline{EF} \times m\overline{HM}$	Area = base x altitude
But $m\overline{AB} = m\overline{EF}$	
$\therefore$ $m\overline{EF} \times m\overline{DL} = m\overline{EF} \times m\overline{HM}$	Dividing by $m\overline{EF}$ we get
$m\overline{DL} = m\overline{HM}$ so altitudes are equal	

Theorem Triangles on the same base and of the same (i.e., equal) altitudes are equal in area.



Given Δs ABC, DBC on the same base $\overline{BC}$ and having equal altitudes.

To Prove Area of (ΔABC) = area of (ΔDBC)

Construction Draw $\overline{BM}$ || to $\overline{CA}$, $\overline{CN}$ || to $\overline{BD}$ meeting $\overline{AD}$ produced in M, N.

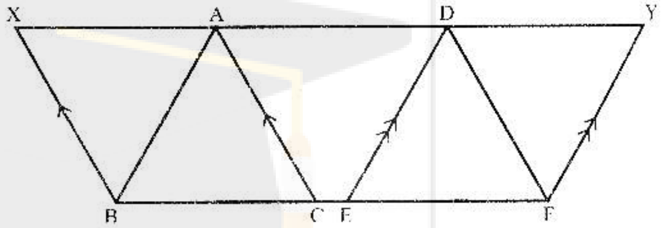
Proof

Statements	Reasons
Δ ABC and Δ DBC are between the same $\|^s$	Their altitudes are equal
Hence MADN is parallel to $\overline{BC}$	
$\therefore$ Area ($\|^{gm}$ BCAM)=Area ($\|^{gm}$ BCND)....(i)	These $\|^{gms}$ are on the same base $\overline{BC}$ and between the same $\|^s$
But $\Delta ABC = \frac{1}{2}$ ($\|^{gm}$ BCAM) .....(ii)	Each diagonal of a $\|^{gm}$ bisects it into two congruent triangles

and $\Delta DBC = \frac{1}{2}$ ($\|_{gm}$ BCND) .....(iii)	
Hence area (Δ ABC) = Area (Δ DBC)	From (i), (ii) and (iii)

Theorem **Triangles on equal bases and of equal altitudes are equal in area.**

Given



Δs ABC, DEF on equal bases $\overline{BC}$, $\overline{EF}$ and having altitudes equal.

To Prove

Area (Δ ABC) = Area (Δ DEF)

Construction

Place the Δs ABC and DEF so that their equal bases $\overline{BC}$ and $\overline{EF}$ are in the same straight line BCEF and their vertices on the same side of it. Draw BX $\|$ to CA and FY $\|$ to ED meeting AD produced in X, Y respectively

Proof

Statements	Reasons
Δ ABC and Δ DEF are between the same parallels	Their altitudes are equal (given)
$\therefore$ XADY is $\|$ to BCEF	
$\therefore$ Area ($\|^{gm}$ BCAX) = Area ($\|^{gm}$ EFYD) .....(i)	These $\|^{gms}$ are on equal bases and between the same parallels
But $\Delta ABC = \frac{1}{2}$ ($\|^{gm}$ BCAX) .....(ii)	Diagonal of a $\|^{gm}$ bisects it
and $\Delta DEF = \frac{1}{2}$ ($\|_{gm}$ EFYD) .....(iii)	
$\therefore$ area (ΔABC) = area (ΔDEF)	From (i), (ii) and (iii)

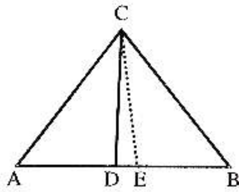
Corollaries

(1) Triangles on equal bases and between the same parallels are equal in area.

(2) Triangles having a common vertex and equal bases in the same straight line, are equal in area.

Exercise 16.2

(1) Show that a median of a triangle divides it into two triangles of equal area.



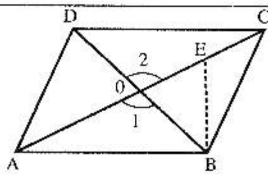
Given Median of the triangle

To Prove: Median divides the triangle into two triangles of equal area.

Proof Make Δ ABC, with $\overline{CD}$ as median and $\overline{CE}$ as altitude

Statements	Reasons
$m\overline{AD} = m\overline{DB}$ ....(i)	D is midpoint of $m\overline{AB}$
Area of the $\Delta ACD = \frac{1}{2} . m\overline{AD}.m\overline{CE}$ ...(ii)	
Area of the $\Delta BCD = \frac{1}{2} . m\overline{BD}.m\overline{CE}$	
$= \frac{1}{2}.m\overline{AD}.m\overline{CE}$ ...(iii)	By (i)
$\Delta ACD = \Delta\ BCD$	By (ii) and (iii)

(2) Prove that a parallelogram is divided by its diagonals into four triangles of equal area.



Given

llgm divided by its diagonals into four triangles

To Prove

Areas of the four triangles arc equal

Construction Make the llgm ABCD with diagonals $m\overline{AC}, m\overline{BD}$ intersecting each other at O. Draw $\overline{BE} \perp \overline{AC}$.

Proof

Statements	Reasons
Area of $\Delta OBC = \frac{1}{2}\ m\overline{OA}.m\overline{BE}$	
$= \frac{1}{2}\ m\overline{OC}.m\overline{BE}$ ....(i)	
The diagonals of the llgm bisect each other	
$\therefore\ m\overline{OA} \cong m\overline{OC}$	
In $\Delta OAB \leftrightarrow \Delta OCD$	
$m\overline{OB} \cong m\overline{OD}$	
$m\overline{OA} \cong m\overline{OC}$	
$<1 \cong\ <2$	opposite angles
$\Delta\ OAB \cong \Delta OCD$ .... (ii)	
$\Delta\ OAD \cong \Delta\ OBC$ ....(iii)	
$\therefore$ Area Δ OAB = Area Δ OBC = Area ΔOCD = Area Δ ODA	By (i), (ii) ,(iii)

(3) Which of the following are true and which are false?

(i) Area of a figure means region enclosed by bounding lines of closed figure. TRUE

(ii) Similar figures have same area. FALSE

(iii) Congruent figures have same area. TRUE

(iv) A diagonal of a parallelogram divides it into two non-congruent triangles. FALSE

(v) Altitude of a triangle means perpendicular from vertex to the opposite side (base). TRUE

(vi) Area of a parallelogram is equal to the product of base and height. TRUE

Q.4 Find the area of the following.

(i)

Area = 6 x 3 = 18cm^2

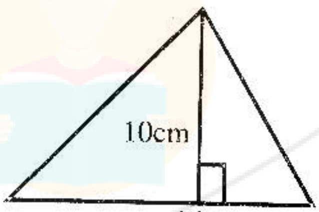
(ii)

Area = 4 x 4 = 16cm^2

(iii)

Area = 8 x 4 = 32cm^2

(iv)



Area = $\frac{1}{2}$ x 16 x 10 = 80cm^2

OBJECTIVE

1. The region enclosed by the bounding lines of a closed figure is called the __ of the figure:
   (a) Area (b) Circle
   (c) Boundary (d) None
2. Base x altitude =
   (a) Area of parallelogram
   (b) Area of square
   (c) Area of Rectangular
   (d) None
3. The union of a rectangular and its interior is called:
   (a) Circle region
   (b) Rectangular region
   (c) Triangle region
   (d) None
4. If a is the side of a square, its area =
   (a) a square unit
   (b) a^2 square units
   (c) a^3 square units
   (d) a^4 square units
5. The union of a triangle and its interior is called as:
   (a) Triangular region
   (b) Rectangular region
   (c) Circle region
   (d) None of these
6. Altitude of a triangle means perpendicular distance to base from its opposite:____
   (a) Vertex (b) Side
   (c) Midpoint (d) None

ANSWER KEY

1.	a	2.	a	3.	b	4.	b	5.	a	6.	a

Unit 17

PRACTICAL GEOMETRY-TRIANGLES

Exercise 17.1

1. Construct a ΔABC, in which:

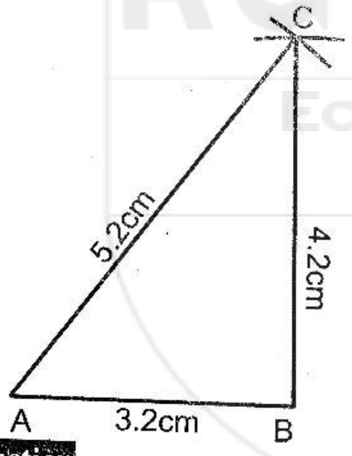
(i) $m\overline{AB} = 3.2cm$, $m\overline{BC} = 4.2cm$, $m\overline{CA} = 5.2cm$

Given

The sides $m\overline{AB} = 3.2cm$, $m\overline{BC} = 4.2cm$, $m\overline{CA} = 5.2cm$ of ΔABC

Required

To construct the ΔABC



Construction

(i) Draw a line segment $m\overline{AB} = 3.2cm$

(ii) With centre B and radius 4.2cm, draw an arc.

(iii) With centre A and radius 5.2cm, draw another arc which meet previous arc at point C.

(iv) Join C to B and A.

Then ABC is the required Δ.

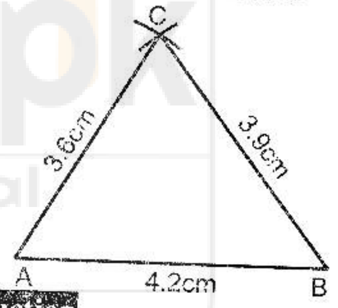
(ii) $m\overline{AB} = 4.2cm, m\overline{BC} = 3.9cm$, $m\overline{CA} = 3.6cm$

Given

The sides $m\overline{AB} = 4.2cm$, $m\overline{BC} = 3.9cm$, $m\overline{CA} = 3.6cm$ of ΔABC

Required

To construct the ΔABC



Construction

(i) Draw a line segment $m\overline{AB} = 4.2cm$

(ii) With centre B and radius 3.9cm, draw an arc.

(iii) With centre A and radius 3.6cm, draw another arc which meet previous arc at point C.

(iv) Join A to C and B to C.

Then ABC is the required Δ.

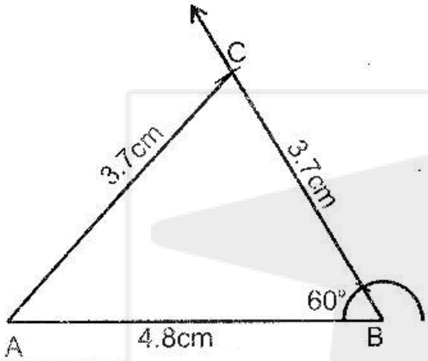
(iii) $m\overline{AB} = 4.8cm$, $m\overline{BC} = 3.7cm$, $m\angle B = 60^\circ$

Given

The sides $m\overline{AB} = 4.8cm$, $m\overline{BC} = 3.7cm$ and $m\angle B = 60^\circ$ of ΔABC

Required

To construct the ΔABC



Construction

(i) Draw a line segment $m\overline{AB} = 4.8cm$

(ii) At the end point B of $\overline{AB}$ make $m\angle B = 60^\circ$.

(iii) Cut off $m\overline{BC} = 3.7cm$ from the terminal side of $\angle 60^\circ$.

(iv) Join AC

Then ABC is the required Δ.

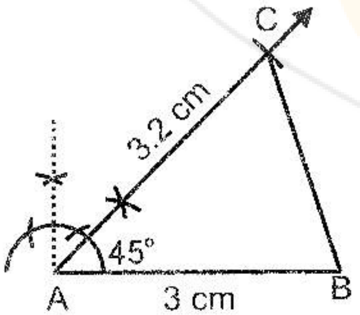
(iv) $m\overline{AB} = 3cm$, $m\overline{AC} = 3.2cm$, $m\angle A = 45^\circ$.

Given

The sides $m\overline{AB} = 3cm$, $m\overline{AC} = 3.2cm$ and $m\angle A = 45^\circ$ of ΔABC

Required

To construct the ΔABC



Construction

(i) Draw a line segment $m\overline{AB} = 3cm$.

(ii) At the end point A of $\overline{AB}$ make $m\angle A = 45^\circ$.

(ii) Cut off $m\overline{AC} = 3.2cm$ from the terminal side of $\angle 45^\circ$.

(iv) Join BC

Then ABC is the required Δ.

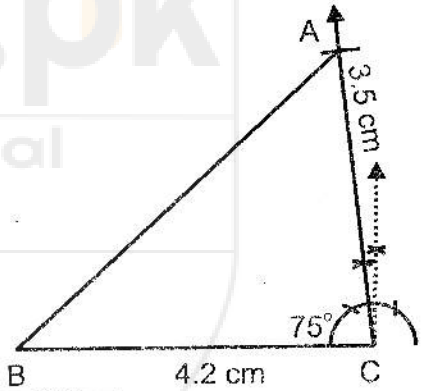
(v) $m\overline{BC} = 4.2cm$, $m\overline{CA} = 3.5cm$, $m\angle C = 75^\circ$

Given

The sides $m\overline{BC} = 4.2cm$, $m\overline{CA} = 3.5cm$ and $m\angle C = 75^\circ$ of ΔABC

Required

To construct the ΔABC



Construction

(i) Draw a line segment $m\overline{BC} = 4.2cm$.

(ii) At the end point C of $\overline{BC}$ make $m\angle C = 75^\circ$.

(iii) Cut off $m\overline{AC} = 3.5cm$ from the terminal side of $\angle 75^\circ$.

(iv) Join AB.

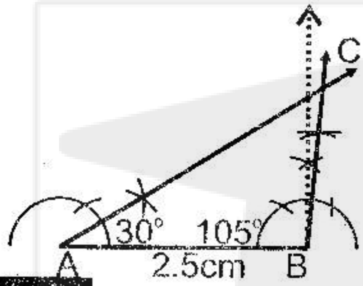
Then ABC is the required Δ.

(vi) $m\overline{AB} = 2.5cm$, $m\angle A = 30^\circ$, $m\angle B = 105^\circ$.

The side $m\overline{AB} = 2.5cm$ and angles $m\angle A = 30^\circ, m\angle B = 105^\circ$ of ΔABC

Required

To construct the ΔABC



Construction

(i) Draw the line segment $m\overline{AB} = 2.5cm$.

(ii) At the end point A of $\overline{AB}$ make $\angle A = 30^\circ$.

(iii) At the end point B of $\overline{AB}$ make $m\angle B = 105^\circ$.

(iv) The terminal sides of these two angles meet in C.

Then ABC is required Δ.

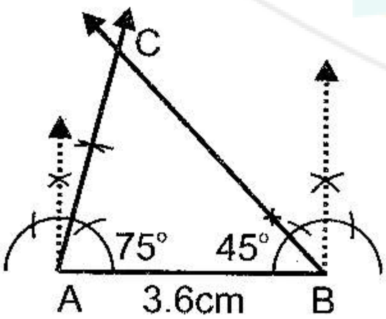
(vii) $m\overline{AB} = 3.6cm, m\angle A = 75^\circ$, $m\angle B = 45^\circ$.

Given

The side $m\overline{AB} = 3.6cm$ and angles $m\angle A = 75^\circ$, $m\angle B = 45^\circ$ of ΔABC

Required

To construct the ΔABC



(i) Draw the line segment $m\overline{AB} = 3.6cm$.

(ii) At the end point A of $\overline{AB}$ make $m\angle A = 75^\circ$.

(iii) At the end point B of $\overline{AB}$ make $m\angle B = 45^\circ$.

(iv) The terminal sides of these two angles meet at C.

Then ABC is the required Δ.

Q.2. Construct a Δ xyz in which

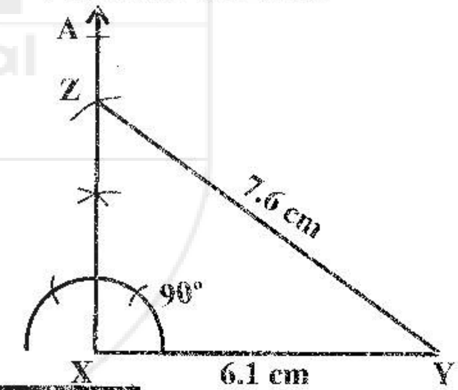
(i) $m\overline{YZ} = 7.6cm, m\overline{XY} = 6.1cm$, $m\angle X = 90^\circ$.

Given

The sides $m\overline{YZ} = 7.6cm, m\overline{XY} = 6.1cm$ and $m\angle X = 90^\circ$ of ΔXYZ.

Required

To construct the ΔXYZ



Construction

(i) Draw the line segment $m\overline{XY} = 6.1cm$

(ii) At the end point X of $\overline{XY}$ make $m\angle X = 90^\circ$.

(iii) With Y as centre and radius 7.6cm, draw an arc which cut terminal side of $\angle 90^\circ$ at point Z.

(iv) Join ZY.

Then XYZ is the required Δ.

(ii) **$m\overline{ZX} = 6.4cm$, $m\overline{YZ} = 2.4cm$,**

$m\angle Y = 90^\circ$

Given

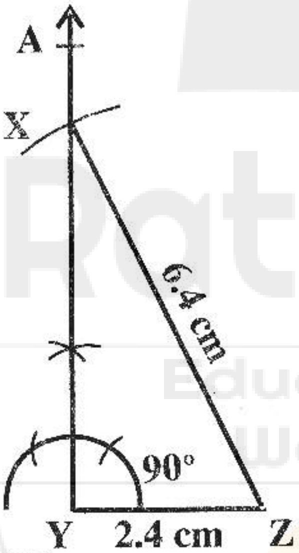
The sides

$m\overline{ZX} = 6.4cm, m\overline{YZ} = 2.4cm$ and

$m\angle Y = 90^\circ$ of **ΔXYZ.**

Required

To construct the ΔXYZ



Construction

(i) Draw the line segment $m\overline{YZ} = 2.4cm$

(ii) At the end point Y of $\overline{YZ}$ make $m\angle Y = 90^\circ$.

(iii) With Z as centre and radius 6.4cm draw an arc which cut terminal side of $\angle 90^\circ$ at point X.

(iv) Join XZ.

Then XYZ is the required Δ.

(iii) $m\overline{XY} = 5.5cm, m\overline{ZX} = 4.5cm$,

$m\angle Z = 90^\circ$

Given

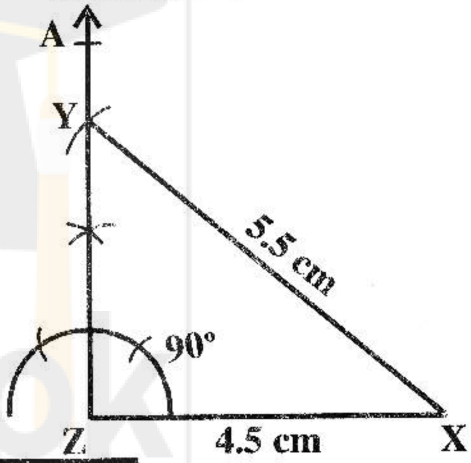
The sides

$m\overline{XY} = 5.5cm, m\overline{ZX} = 4.5cm$ and

$m\angle Z = 90^\circ$ of ΔXYZ.

Required

To construct the ΔXYZ



Construction

(i) Draw a line segment $m\overline{ZX} = 4.5cm$

(ii) At the end point Z of $\overline{ZX}$ make $m\angle Z = 90^\circ$.

(iii) With X as centre and radius 5.5cm draw an arc which cut terminal side of $\angle 90^\circ$ at point Y

(iv) Join XY.

Then XYZ is the required Δ.

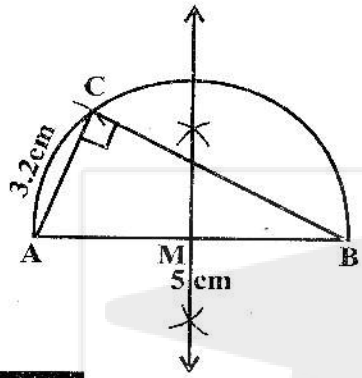
Q.3. Construct a right angled Δ measure of whose hypotenuse is 5cm and one side is 3.2cm.

Given

In right angled **Δ** hypotenuse is 5cm and one side is 3.2cm

Required

To construct the ΔXYZ



Construction

(i) Draw a line segment $m\overline{AB} = 5cm$.

(ii) With $\overline{AB}$ as diameter, draw a semi circle.

(iii) With A as center draw an arc of radius 3.2cm cutting the semi circle in C.

(iv) Join C with A and B.

Therefore ABC is required triangle with $\angle C=90^\circ$

Q.4 Construct a right angled isosceles triangle. Whose hypotenuse is:

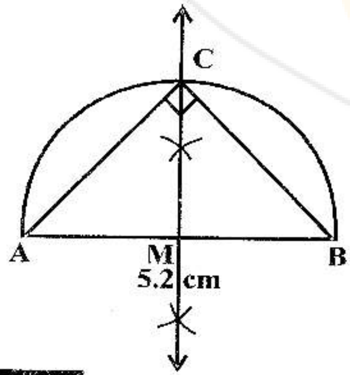
i) Hypotenuse 5.2cm long

Given

In right angled isosceles triangle hypotenuse is 5.2 cm.

Required

To construct right angled isosceles triangle



Construction

(i) Take $m\overline{AB} = 5.2cm$.

(ii) Find mid-point M of $\overline{AB}$.

(iii) With centre as M and radius $m\overline{AM} = m\overline{MB}$ draw a semi circle which intersects the bisector in C.

(iv) Join A to C and B to C.

Then ΔABC is the required right angled isosceles triangle with $\angle C = 90^\circ$

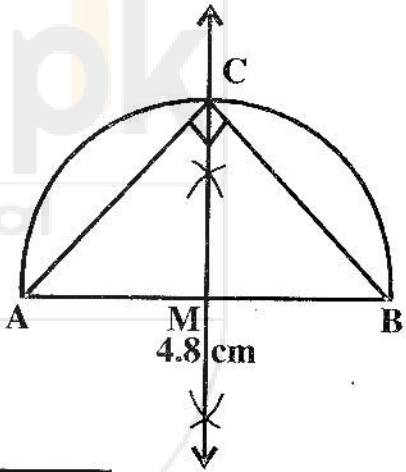
(ii) Hypotenuse 4.8 cm

Given

In right angled isosceles triangle hypotenuse is 4.8 cm.

Required

To construct right angled isosceles triangle.



Construction

(i) Take $m\overline{AB} = 4.8cm$.

(ii) Find mid-point M of $\overline{AB}$.

(iii) With centre as M and radius $m\overline{AM} = m\overline{MB}$ draw a semi circle which intersects the bisector in C.

(iv) Join A to C and B to C.

Then ΔABC is the required right angled isosceles triangle with $\angle C = 90^\circ$

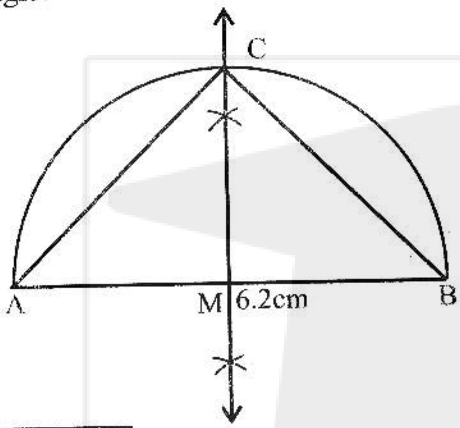
(iii) Hypotenuse 6.2 cm

Given

In right angled isosceles triangle hypotenuse is 6.2 cm.

Required

To construct right angled isosceles triangle.



Construction

(i) Take $m\overline{AB} = 6.2cm$.

(ii) Find mid-point M of AB.

(iii) With centre as M and radius $m\overline{AM} = m\overline{MB}$ draw a semi circle which intersects the bisector in C.

(iv) Join A to C and B to C.

Then ΔABC is the required right angled isosceles triangle with $\angle C = 90^{\circ}$

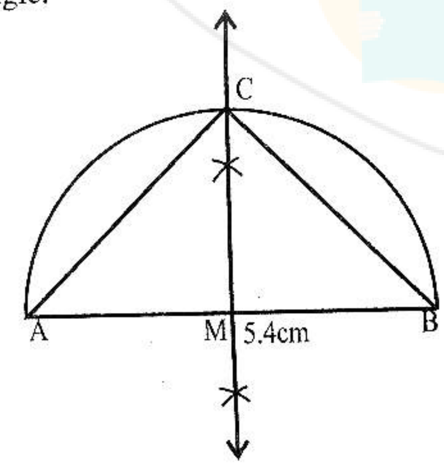
(iv) Hypotenuse 5.4 cm

Given

In right angled isosceles triangle hypotenuse is 5.4 cm.

Required

To construct right angled isosceles triangle.



Construction

(i) Take $m\overline{AB} = 5.4cm$.

(ii) Find mid-point M of AB.

(iii) With centre as M and radius $m\overline{AM} = m\overline{MB}$ draw a semi circle which intersects the bisector in C.

(iv) Join A to C and B to C.

Then ΔABC is the required right angled isosceles triangle with $\angle C = 90^{\circ}$

Q.5.(Ambiguous case) construct a ΔABC in which

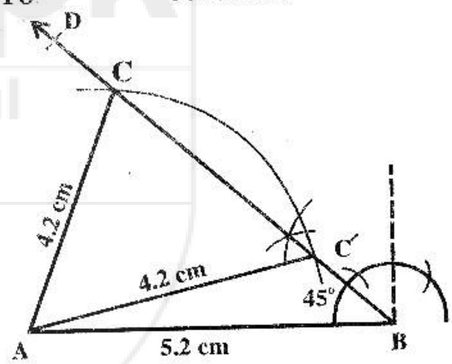
(i) $m\overline{AC} = 4.2cm, m\overline{AB} = 5.2cm$, $m\angle B = 45^{\circ}$.

Given

In ΔABC $m\overline{AC} = 4.2cm, m\overline{AB} = 5.2cm$, $m\angle B = 45^{\circ}$.

Required

To construct ΔABC



Construction

(i) Draw a line segment $m\overline{AB} = 5.2cm$.

(ii) At the end point B of $\overline{BA}$ make $m\angle B = 45^{\circ}$.

(iii) With centre A and radius 4.2cm draw an arc which cuts $\overline{BD}$ in two distinct points C and C'.

(iv) Join AC and AC'.

$\therefore$ ΔABC and $\Delta ABC'$ are required triangles.

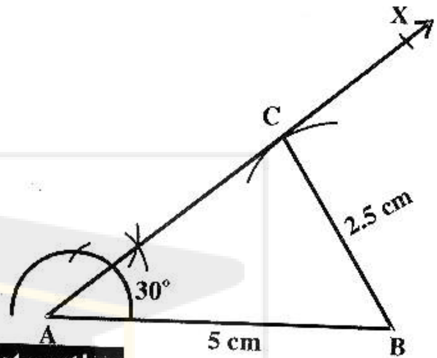
(ii) $\mathbf{m\overline{BC} = 2.5cm, m\overline{AB} = 5.0cm,}$ $\mathbf{m\angle A = 30^o.}$

Given

In ΔABC $m\overline{BC} = 2.5cm$, $m\overline{AB} = 5.0cm$, $m\angle A = 30^o$.

Required

To construct ΔABC



Construction

(i) Take $m\overline{AB} = 5cm$.

(ii) At the end point A of $\overline{AB}$ make $m\angle A = 30^o$.

(iii) With centre B and radius 2.5cm draw an arc which touches $\overrightarrow{AX}$ at point C.

(iv) Join BC.

$\therefore$ ΔABC is required triangle.

Exercise 17.2

1. Construct the following Δ's ABC. Draw the bisectors of their angles and verify their concurrency.

(i) $\mathbf{m\overline{AB} = 4.5cm, m\overline{BC} = 3.1cm,}$ $\mathbf{m\overline{CA} = 5.2cm.}$

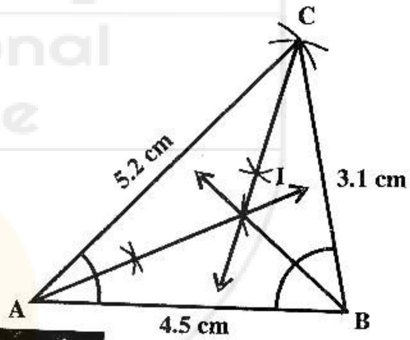
Given

The sides $m\overline{AB} = 4.5cm$, $m\overline{BC} = 3.1cm$, and $m\overline{CA} = 5.2cm$.

Required

(i) To construct ΔABC.

(ii) To draw its angle bisectors and verify their concurrency.



Construction

(i) Take $m\overline{AB} = 4.5cm$.

(ii) With A as centre and radius 5.2cm draw an arc.

(iii) With B as centre and radius 3.1cm draw another arc which intersect the first arc at C.

(iv) Join AC and BC to complete the ΔABC.

(v) Draw bisectors of $\angle A, \angle B$ and $\angle C$ meeting each other in the point I.

Hence angle bisectors of the ΔABC are concurrent at I which lies within the triangle.

(ii) $\mathbf{m\overline{AB} = 4.2cm, m\overline{BC} = 6cm,}$ $\mathbf{mCA = 5.2cm}$

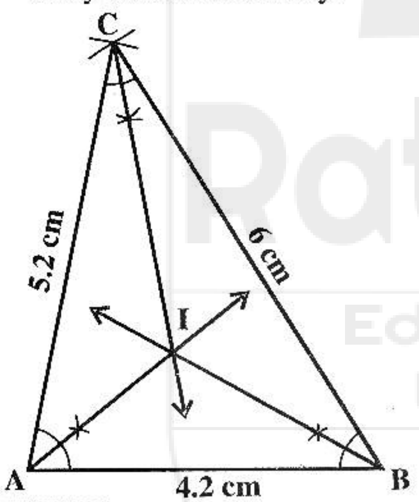
Given

The sides $m\overline{AB} = 4.2cm$, $m\overline{BC} = 6cm$, $m\overline{CA} = 5.2cm$ of a ΔABC.

Required

(i) To construct ΔABC.

(ii) To draw its angle bisectors and verify their concurrency.



Construction

(i) Take $m\overline{AB} = 4.2cm$.

(ii) With A as centre and radius 5.2cm draw an arc.

(iii) With B as centre and radius 6cm draw another arc which intersect the first arc at C.

(iv) Join BC and AC to complete the ΔABC.

(v) Draw bisectors of $\angle A, \angle B$ and $\angle C$ meeting each other in the point I . Hence angle bisectors of the ΔABC are concurrent at I which lies within the triangle.

(iii) $\mathbf{m\overline{AB} = 3.6cm, m\overline{BC} = 4.2cm,}$ $\mathbf{m\angle B = 75^o.}$

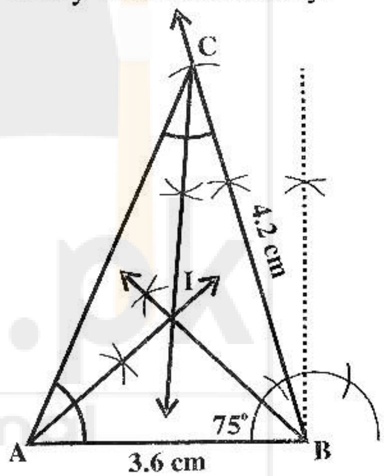
Given

The sides $m\overline{AB} = 3.6cm$, $m\overline{BC} = 4.2cm$ and $m\angle B = 75^o$ of ΔABC

Required

(i) To construct ΔABC.

(ii) To draw its angle bisectors and verify their concurrency.



Construction

(i) Take $m\overline{AB} = 3.6cm$.

(ii) At B draw angle of 75^o

(iii) With B as centre and radius 4.2cm draw arc which intersect terminal arm of 75^o in C.

(iv) Join AC to complete the ΔABC.

(v) Draw bisectors of $\angle A, \angle B$ and $\angle C$ meeting each other in the point I.

Hence angle bisectors of the ΔABC are concurrent at I which lies within the triangle.

Q.2. Construct Δs PQR. Draw their altitudes and show that they are concurrent.

(i) $m\overline{PQ} = 6cm, m\overline{QR} = 4.5cm, m\overline{PR} = 5.5cm.$

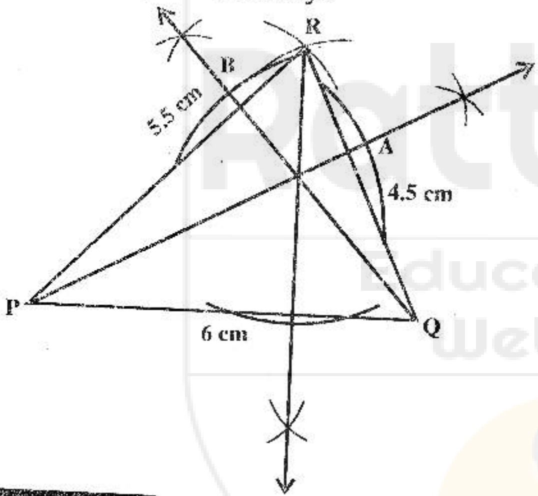
Given

The sides $m\overline{PQ} = 6cm$, $m\overline{QR} = 4.5cm$ and $m\overline{PR} = 5.5cm$ of a ΔPQR.

Required

(i) To construct Δ PQR.

(ii) To draw its altitudes and verify their concurrency.



Construction

(i) Take $m\overline{PQ} = 6cm$

(ii) With P as centre draw an arc of radius 5.5 cm.

(iii) With Q as centre draw an arc of radius 4.5cm, cutting the first in R.

(iv) Join R with P and Q.

(v) Draw the altitudes on, $\overline{PR}, \overline{QR}$ and $\overline{PQ}$ which cut each other in I.

(vi) All altitudes are concurrent.

(ii) $m\overline{PQ} = 4.5cm, m\overline{QR} = 3.9cm, m\angle R = 45°.$

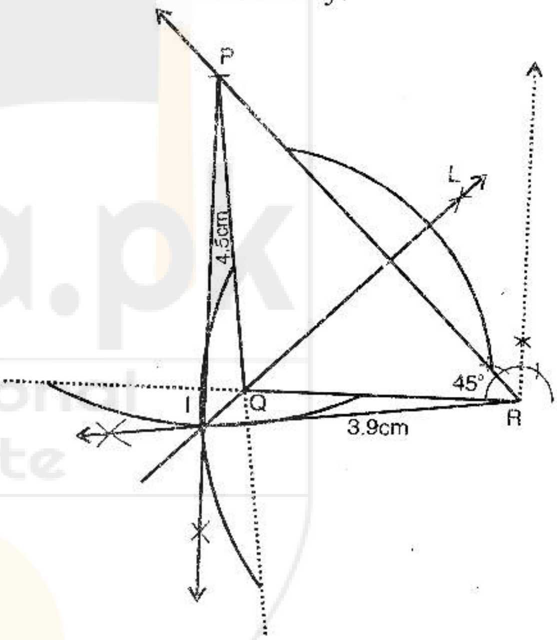
Given

The sides $m\overline{PQ} = 4.5cm$, $m\overline{QR} = 3.9cm$ and $m\angle R = 45°$ of ΔPQR

Required

(i) To construct Δ PQR.

(ii) To draw its altitudes and verify their concurrency.



Construction

(i) Draw $\overline{QR} = 3.9cm$.

(ii) Make $\angle R = 45°$

(iii) Cut $\overline{QP} = 4.5cm$ join PQ, ΔPQR is formed.

(iv) Draw altitudes on $\overline{PR}$, $\overline{QR}$ and $\overline{PQ}$ they cut each other in I.

The altitudes are concurrent.

(iii) $\mathbf{m\overline{RP} = 3.6cm, m\angle Q = 30^o}$,

$\mathbf{m\angle P = 105^o}$.

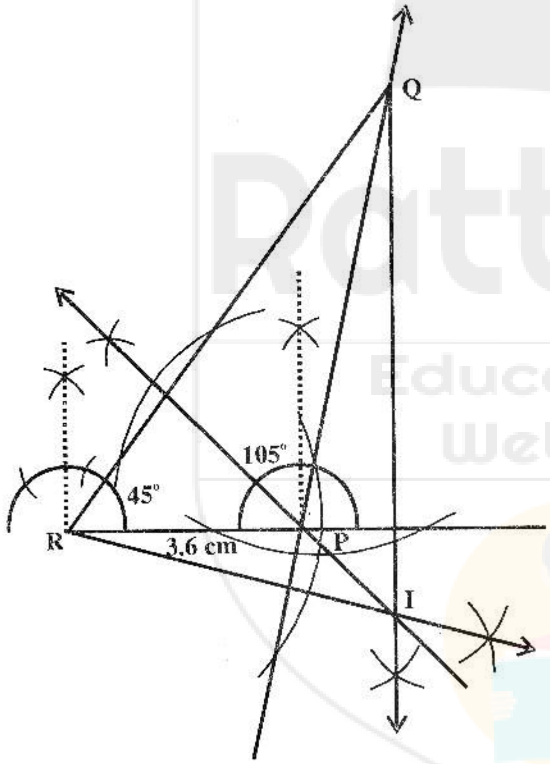
Given

$m\overline{RP} = 3.6cm, m\angle Q = 30^o$, $m\angle P = 105^o$.

Required

(i) To construct Δ PQR.

(ii) To draw its altitudes and verify their concurrency.



Construction

$$m\angle P + m\angle Q + m\angle R = 180^o$$

$$105^o + 30^o + m\angle R = 180^o$$

$$135^o + m\angle R = 180^o$$

$$m\angle R = 180^o - 135^o = 45^o$$

(i) Take $m\overline{RP} = 3.6cm$.

(ii) At P draw an angle of 105^o.

(iii) At R draw an angle of 45^o.

(iv) Terminal arms of both angles meet in point Q. It form Δ PQR.

(v) Draw the altitudes, of $\overline{PQ}$ and $\overline{QR}$ and $\overline{RP}$ cutting each other in I.

The altitudes are concurrent.

Q.3. Construct the following triangles ABC. Draw the perpendicular bisectors of their sides and verify their concurrency. Do they meet inside the triangle.

(i) $\mathbf{m\overline{AB} = 5.3cm}$, $\mathbf{m\angle A = 45^o}$,

$\mathbf{m\angle B = 30^o}$

Given

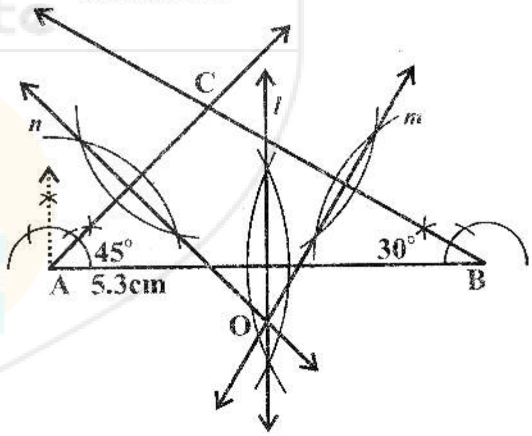
Side $m\overline{AB} = 5.3cm$ and $m\angle A = 45^o$,

$m\angle B = 30^o$ of a ΔABC.

Required

(i) To construct the ΔABC.

(ii) To draw perpendicular bisectors of its sides and to verify that they are concurrent.



Construction

(i) Take $m\overline{AB} = 5.3cm$

(ii) At the end point A of $\overline{AB}$ make $m\angle A = 45^o$.

(iii) At the end point B of $\overline{AB}$ make $m\angle B = 30^{o}$.

(iv) The terminal sides of these two angles meet at C.

Then ABC is required Δ.

(v) Draw perpendicular bisectors of $\overline{AB}, \overline{BC}$ and $\overline{CA}$ meeting each other in the point O.

Hence the three perpendicular bisectors of sides of ΔABC are concurrent at O.

(ii) $m\overline{BC} = 2.9cm$, $m\angle A = 30^{o}$, $m\angle B = 60^{o}$

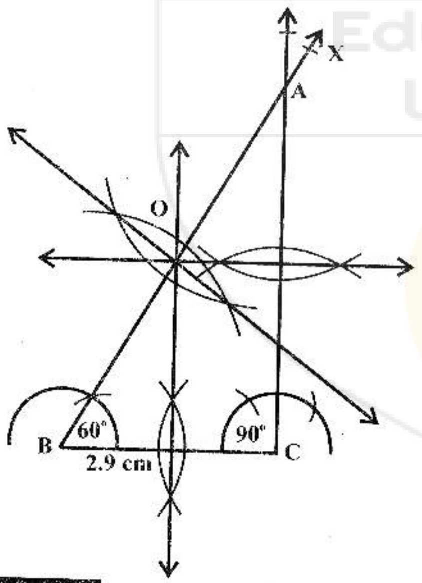
Given

The side $m\overline{BC} = 2.9cm$, $m\angle A = 30^{o}$ and $m\angle B = 60^{o}$ of ΔABC.

Required

(i) To construct the ΔABC.

(ii) To draw perpendicular bisectors of its sides and to verify that they are concurrent.



Construction

$$m\angle A + m\angle B + m\angle C = 180^{o}$$

$$30^{o} + 60^{o} + m\angle C = 180^{o}$$

$$90^{o} + m\angle C = 180^{o}$$

$$m\angle C = 90^{o}.$$

(i) Take $m\overline{BC} = 2.9cm$

(ii) At the end point B of $\overline{BC}$ make $m\angle B = 60^{o}$.

(iii) At the end point C of $\overline{BC}$ make $m\angle C = 90^{o}$.

(iv) The terminal sides of these two angles meet in A.

Then ABC is required Δ.

(v) Draw perpendicular bisectors of $\overline{AB}, \overline{BC}$ and $\overline{CA}$ meeting each other in the point O.

Hence the three perpendicular bisectors of sides of ΔABC are concurrent at O.

(iii) $m\overline{AB} = 2.4cm$, $m\overline{AC} = 3.2cm$, $m\angle A = 120^{o}$

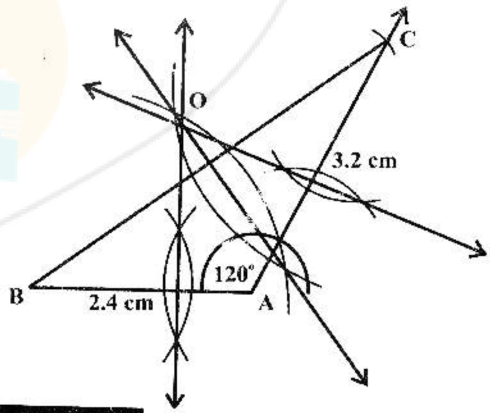
Given

The sides $m\overline{AB} = 2.4cm$, $m\overline{AC} = 3.2cm$ $m\angle A = 120^{o}$ of a ΔABC

Required

(i) To construct the ΔABC.

(ii) To draw perpendicular bisectors of its sides and to verify that they are concurrent.



Construction

(i) Take $m\overline{AB} = 2.4cm$

(ii) At the end point A of $\overline{AB}$ make $m\angle A = 120^\circ$.

(iii) With centre A, draw an arc of radius 3.2cm which cut terminal arm of $\angle A$ at C.

(iv) Join B to C

Then ABC is required Δ.

(v) Draw perpendicular bisectors of $\overline{AB}, \overline{BC}$ and $\overline{CA}$ meeting each other at the point O.

Hence the three perpendicular bisectors of sides of ΔABC are concurrent at O.

Q.4. Construct following Δ's XYZ. Draw their three medians and show that they are concurrent.

(i) $m\overline{YZ} = 4.1cm$, $m\angle Y = 60^\circ$ and $m\angle X = 75^\circ$

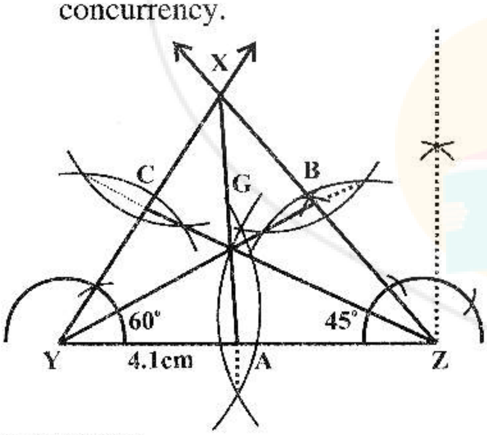
Given

The side $m\overline{YZ} = 4.1cm$, $m\angle Y = 60^\circ$ and $m\angle X = 75^\circ$

Required

(i) Construct the ΔXYZ.

(ii) Draw its medians and verify their concurrency.



Construction

$$m\angle X + m\angle Y + m\angle Z = 180^\circ$$

$$75^\circ + 60^\circ + m\angle Z = 180^\circ$$

$$135^\circ + m\angle Z = 180^\circ$$

$$m\angle Z = 180^\circ - 135^\circ$$

$$m\angle Z = 45^\circ.$$

(i) Take $m\overline{YZ} = 4.1cm$.

(ii) At the end point y of $\overline{YZ}$ make $m\angle Y = 60^\circ$.

(iii) At the end point Zof $\overline{ZY}$ make $m\angle Z = 45^\circ$

(iv) The terminal sides of these angles meet at X. Then XYZ is required Δ.

(v) Draw perpendicular bisectors of the sides $\overline{YZ}$, $\overline{ZX}$ and $\overline{XY}$ of ΔXYZ and make their midpoints A, B and C respectively.

(vi) Join Y to midpoint B to get median $\overline{YB}$.

(vii) Join Z to midpoint C to get median $\overline{ZC}$.

(viii) Join X to mid point A to get median $\overline{AX}$.The medians of ΔXYZ pass through the same point G.

All medians intersect at point G.

Hence medians are concurrent at G.

(ii) $m\overline{XY} = 4.5cm, m\overline{YZ} = 3.4cm$, $m\overline{ZX} = 5.6cm$

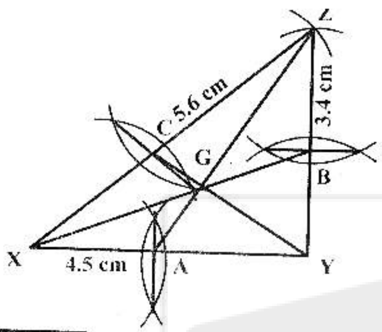
Given

The sides $m\overline{XY} = 4.5cm$, $m\overline{YZ} = 3.4cm$ and $m\overline{ZX} = 5.6cm$ of a ΔXYZ.

Required

(i) Construct the ΔXYZ.

(ii) Draw its medians and verify their concurrency.



Construction

(i) Take $m\overline{XY} = 4.5cm$.

(ii) With Y as centre and radius 3.4 cm draw an arc.

(iii) With X as centre and radius 5.6cm draw another arc cutting first in Z join Z to Y and X to Z.

(iv) Draw perpendicular bisectors of the sides $\overline{XY}$, $\overline{YZ}$ and $\overline{XZ}$ of ΔXYZ and make their midpoints A,B and C respectively.

(v) Join X to mid point B to get median $\overline{XB}$.

(vi) Join Y to midpoint C to get medians $\overline{YC}$.

(vii) Join Z to midpoint A to get median $\overline{ZA}$.

All medians intersect at point G.

Hence medians are concurrent at G.

(iii) $m\overline{ZX} = 4.3cm, m\angle X = 75^o$ and $m\angle Y = 45^o$.

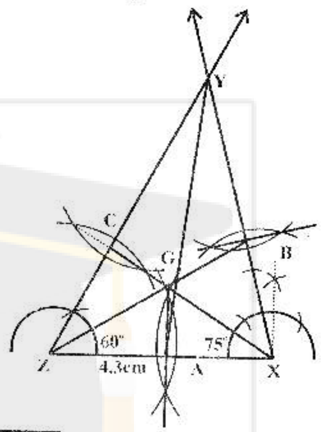
Given

The side $m\overline{ZX} = 4.3cm$, $m\angle X = 75^o$ and $m\angle = 45^o$ of ΔXYZ.

Required

(i) Construct the ΔXYZ.

(ii) Draw its medians and verify their concurrency.



Construction

$$m\angle X + m\angle Y + m\angle Z = 180^o$$
$$75^o + 45^o + m\angle Z = 180^o$$
$$m\angle Z + 120^o = 180^o$$
$$m\angle Z = 180^o - 120^o$$
$$m\angle Z = 60^o$$

(i) Take $m\overline{ZX} = 4.3cm$.

(ii) At the end point Z of $\overline{ZX}$ make $m\angle Z = 60^o$.

(iii) At the end point X of $\overline{XY}$ make $m\angle X = 75^o$

(iv) The terminal sides of these angles meet at Y. Then XYZ is required Δ.

(v) Draw perpendicular bisectors of the sides $\overline{ZX}$, $\overline{XY}$ and $\overline{YZ}$ of ΔXYZ and make their midpoints A,B and C respectively.

(vi) Join Y to midpoint A to get median $\overline{YA}$.

(vii) Join Z to the midpoint B to get median $\overline{ZB}$.

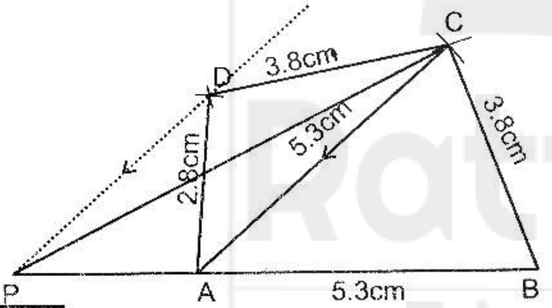
(viii) Join X to the midpoint B to get median $\overline{XC}$.

All medians intersect at point G.
Hence medians are concurrent at G.

Exercise 17.3

1. (i) Construct a quadrilateral ABCD, having $m\overline{AB} = m\overline{AC} = 5.3cm$, $m\overline{BC} = m\overline{CD} = 3.8cm$ and $m\overline{AD} = 2.8cm$.

(ii) On the side BC construct a Δ equal in area to the quadrilateral ABCD.



Given

Sides of quadrilateral ABCD

$m\overline{AB} = m\overline{BC} = 5.3$ cm

$m\overline{BC} = m\overline{CD} = 3.8$ cm

$m\overline{AD} = 2.8$ cm

Required

i) To make the quadrilateral ABCD.

ii) On the side $\overline{BC}$ construct a Δ equal in area to the quadrilateral ABCD.

Construction

(i) Take $m\overline{AB}$ = 5.3 cm.

(ii) With centre A and B draw arcs with radii 5.3 cm and 3.8 cm respectively cutting each other in C.

(iii) With C as centre draw an arc of radius 3.8cm, then with A as centre draw an arc of radius 2.8cm cutting the first in D.

(iv) Join AD, DC, BC

ABCD is the required quadrilateral.

(ii)

(i) Draw $\overline{AC}$

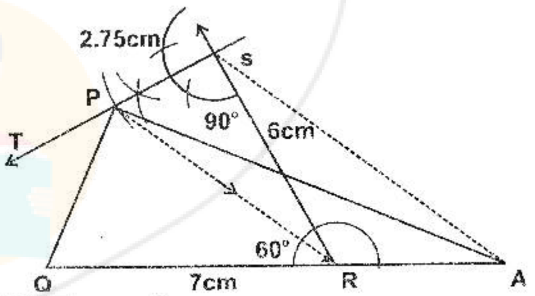
(ii) Through D draw a line || $\overline{AC}$

(iii) Produce $\overline{AB}$ which meet parallel line in P.

(iv) Join P with C

PCB is the required triangle equal in area to quadrilateral ABCD.

2. Construct a Δ equal in area to the quadrilateral PQRS, having $m\overline{QR} = 7cm$, $m\overline{RS} = 6cm$, $m\overline{SP} = 2.75cm$, $m\angle QRS = 60^\circ$ and $m\angle RSP = 90^\circ$.



Given

Parts of the quadrilateral PQRS are given.

Required

(i) To make the quadrilateral PQRS.

(ii) To make a Δ equal in area to the quadrilateral PQRS.

Construction

(i) Take $m\overline{QR} = 7cm$

(ii) Make $\angle QRS = 60^o$

(iii) With R as centre draw an arc of 6 cm radius which cuts terminal arm of $\angle 60^o$ in S.

(iv) Make $\angle RSP = 90^o$

(v) With S as centre draw an arc of 2.75 cm radius which cuts terminal arm of 90^o in P.

(vi) Join QP.

PQRS is required quadrilateral.

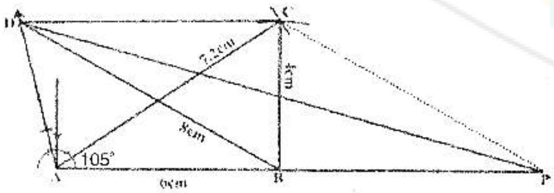
(vii) Join PR

(viii) Through S draw a line parallel to $\overline{PR}$ which meet $\overline{QR}$ produced in A.

(ix) Join AP.

ΔAPQ is the required triangle equal in area to quadrilateral PQRS

3. Construct a Δ equal in area to the quadrilateral ABCD, having $m\overline{AB} = 6cm$, $m\overline{BC} = 4cm$, $m\overline{AC} = 7.2cm$, $m\angle BAD = 105^o$ and $m\overline{BD} = 8cm$.



Given

Parts of the quadrilateral ABCD are given

Required

(i) To make the quadrilateral ABCD.

(ii) To make a Δ with area equal to that of quadrilateral ABCD.

Construction

(i) Take $m\overline{AB} = 6cm$.

(ii) Make $\angle A = 105^o$.

(iii) With B as centre draw an arc of radius 8cm, cutting the arm of $\angle A$ in D.

(iv) With A as centre draw an arc of radius 7.2cm, with B as centre draw an arc of radius 4cm, cutting the first in C. Join C with B and D.

ABCD is the required quadrilateral.

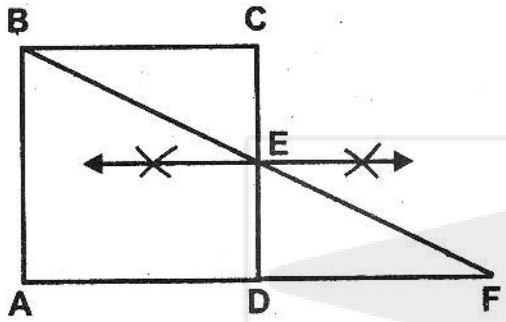
(v) Join AC.

(vi) Join DB. Draw a line parallel to $\overline{DB}$ which meet $\overline{AB}$ produced in P.

(vii) Join PD.

ΔADP is the required triangle equal in area to the quadrilateral ABCD.

4. Construct a right-angled triangle equal in area to a given square.



Given

Square ABCD

Required

To make a right-angle Δ equal in area to the square.

Construction

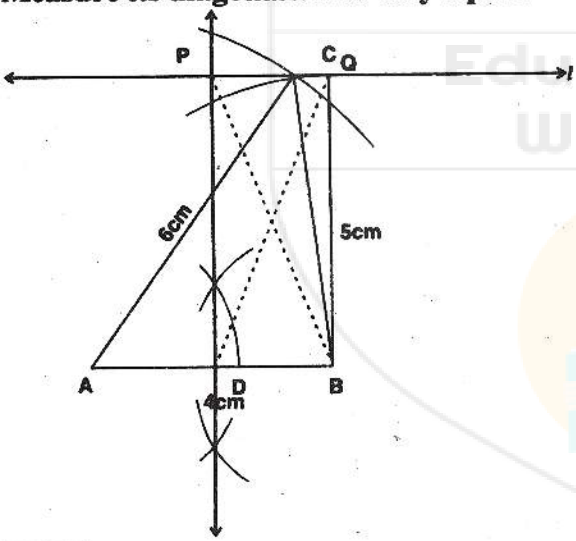
(i) Bisect $\overline{CD}$ at E.

(ii) Join BE and produce it to meet $\overline{AD}$ produced in F.

ΔABF is the required triangle equal in area to square ABCD.

Exercise 17.4

1. Construct a Δ with sides 4 cm, 5 cm and 6 cm and construct a rectangle having its area equal to that of the Δ. Measure its diagonals. Are they equal?



Given

4cm, 5cm, 6cm the sides of the triangle Δ.

Required

To make a rectangle with area equal to that of the Δ.

Construction

(i) Draw $\overline{AB} = 4cm$.

(ii) Draw an arc of radius 5cm with centre B and an other arc of radius 6cm with centre A cutting the first in C.

(iii) Join CA, CB

(iv) ABC is the required Δ.

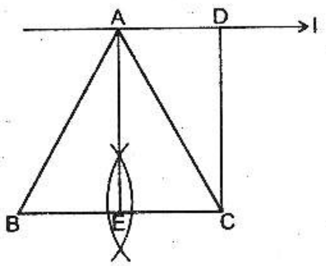
(v) Draw a line ℓ through C $\parallel$ $\overline{AB}$.

(vi) Draw the $\perp$ bisector of $\overline{AB}$ in D and cutting the line ℓ at P.

(vii) Draw BQ $\perp$ on the line ℓ.

PQDB is the required rectangle.

2. Transform an isosceles Δ into a rectangle.



Construction

(i) Take a line $\overline{BC}$

(ii) Draw the $\perp$ bisector of $\overline{BC}$ take any point A on it.

(iii) Join AB and AC.

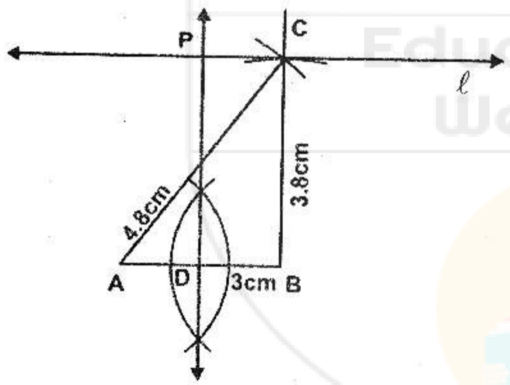
(iv) ΔABC is the isosceles Δ with $m\overline{AB} = m\overline{AC}$.

(v) Through A draw a line $\ell \parallel BC$.

(vi) Draw $\overline{CD} \perp \ell$

CDAE is the required rectangle equal in area to ΔABC

3. Construct a ΔABC such that $m\overline{AB}=3cm$, $m\overline{BC}=3.8cm$, $m\overline{AC}=4.8cm$. Construct a rectangle equal in area to the ΔABC, and measure its sides.



Given

Three sides of the ΔABC

Required

To construct a rectangle with area equal to that of the ΔABC.

Construction

(i) Take $m\overline{AB} = 3cm$

(ii) With B as centre draw an arc of radius 3.8cm, with A as centre draw another arc of radius 4.8cm, cutting the first in C.

(iii) Join B with C and A.

(iv) ABC is the required Δ.

(v) Through C draw a line $\ell \parallel \overline{AB}$.

(vi) Draw the $\perp$ bisector of $\overline{AB}$ cutting the line ℓ in P.

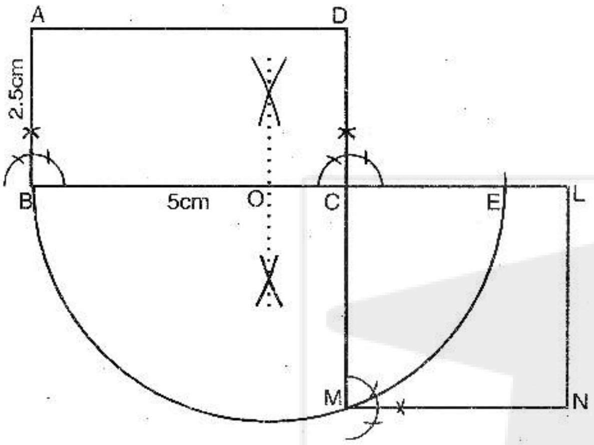
(vii) PCDB is the required rectangle.

Measures of sides of rectangle PCDB are;

$m\overline{PD} = 3.8cm$, $m\overline{DB} = 1.5cm$

Exercise 17.5

1. Construct a rectangle whose adjacent sides are 2.5 cm and 5cm respectively. Construct a square having area equal to the given rectangle.



Construction

(i) Make the rectangle ABCD with given lengths of sides.

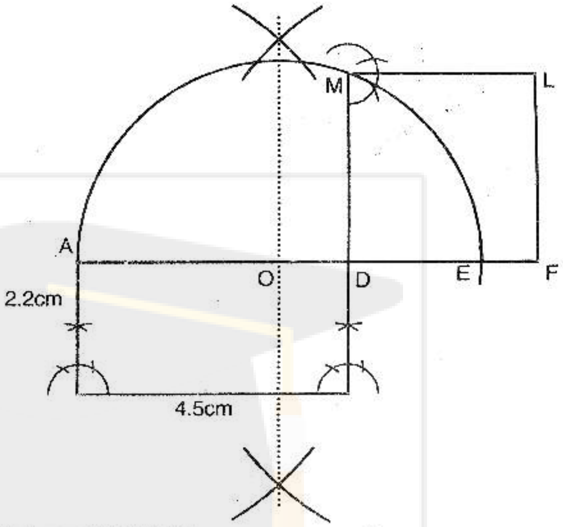
(ii) Produce $\overline{BC}$ and cut $m\overline{CE} = m\overline{CD}$

(iii) Bisect $\overline{BE}$ at O.

(iv) With O as centre and $\overline{OB}$ radius draw a semicircle cutting $\overline{DC}$ produced in M.

(v) With $\overline{CM}$ as side complete the square CMNL.

2. Construct a square equal in area to a rectangle whose adjacent sides are 4.5 cm and 2.2 cm respectively. Measure the sides of the square and find its area and compare with the area of the rectangle.



Construction

(i) Make the rectangle ABCD with given sides.

(ii) Produce AD and cut $m\overline{DE} = m\overline{DC}$.

(iii) Bisect $\overline{AE}$ at O.

(iv) With O as centre and $\overline{OA}$ radius draw a semicircle cutting $\overline{CD}$ produced in M.

(v) With $\overline{DM}$ as side complete the square DFLM.

(vi) Side of the square (average) = 3.15cm

Area = $3.15 \times 3.15 = 9.9\text{cm}^2$

Area of the rectangle $= 2.2 \times 4.5$

$= 9.9\text{cm}^2$ (equal to area of square)

3. In Q.2 above verify by measurement that the perimeter of the square is less than that of the rectangle.

Solution

(i) Side of the square = 3.15cm

Perimeter $P_1 = 4 \times 3.15$

$= 12.60$ cm

Sides of the rectangle are 4.5cm, 2.2cm

Perimeter $P_2 = 2(4.5 + 2.2)$

$= 2(6.7)$

$= 13.4\text{cm}$

$P_1 < P_2$ verified

4. Construct a square equal in area to the sum of two squares having sides 3 cm and 4 cm respectively.

Construction

(i) Make a right angled ΔABC with $\overline{AC} = 3cm$, $\overline{BC} = 4cm$.

(ii) Using Pythagoras theorem

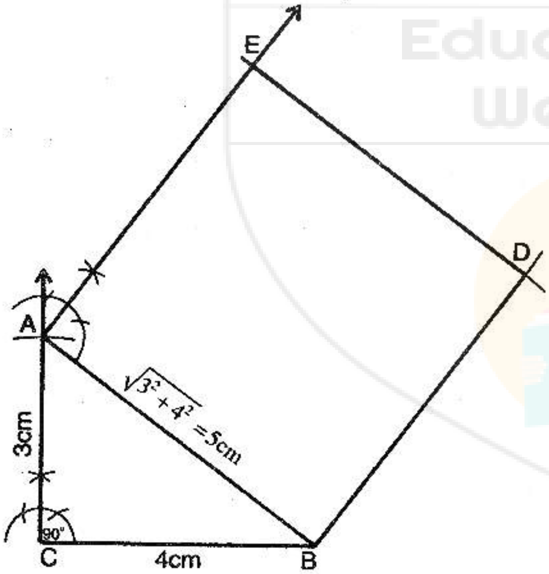
$$\sqrt{|AC|^2 + |BC|^2} = \sqrt{|AB|^2}$$

$$\sqrt{(3)^2 + (4)^2} = \sqrt{|AB|^2}$$

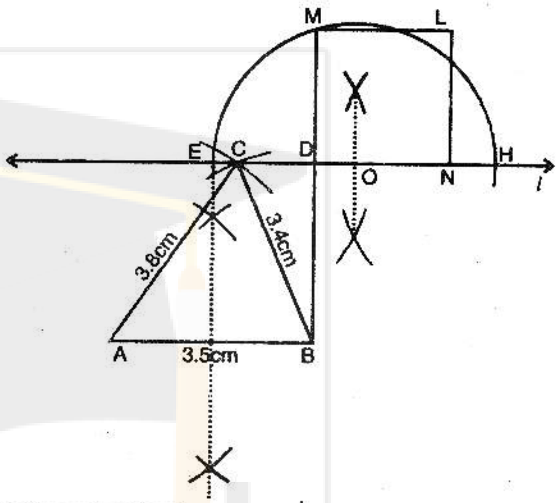
$$5cm = |AB|$$

(ii) With $\overline{AB}$ as side make square ABDE.

(iii) ABDE is the required area of square equal in area to the sum of the areas of two squares.



5. Construct a Δ having base 3.5 cm and other two sides equal to 3.4 cm and 3.8 cm respectively. Transform it into a square of equal area.



Construction

(i) Make the ΔABC with the given sides.

(ii) Draw the $\perp$ bisector of $\overline{AB}$ and a line ℓ through C $\parallel$ $\overline{AB}$ cutting each other in E.

(iii) Draw $\overline{BD} \perp \ell$.

(iv) BDEF is a rectangle.

(v) Produce $\overline{ED}$, cut $\overline{DH} = \overline{DB}$.

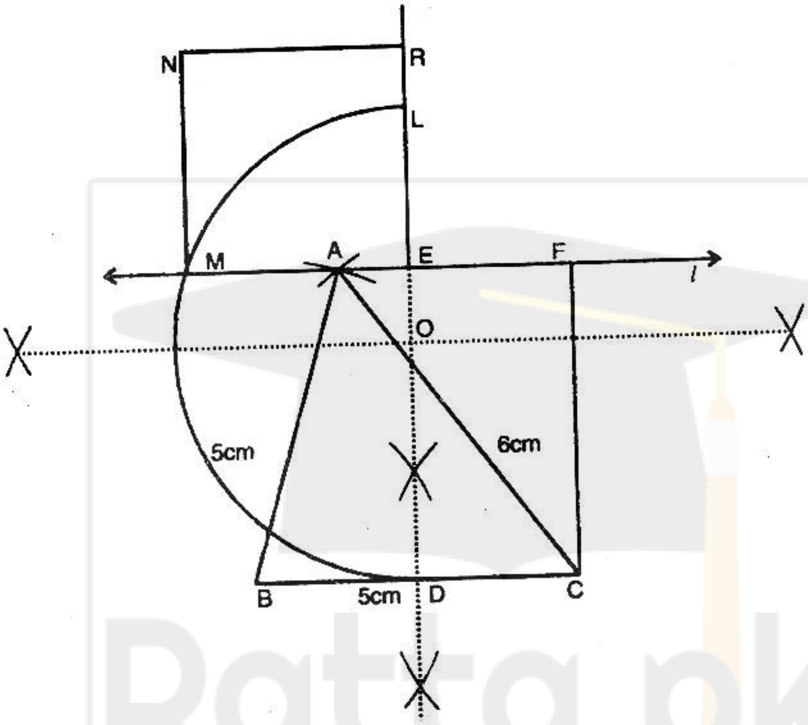
(vi) Bisect $\overline{EH}$ at O.

(vii) With O as centre and $\overline{OE}$ radius draw a semicircle cutting $\overline{BD}$ produced in M.

(viii) With $\overline{DM}$ as side, complete the square DNLM.

This is the required square equal in area to ΔABC.

6. Construct a Δ having base 5 cm and other sides equal to 5 cm and 6 cm. Construct a square equal in area to given Δ.



Construction

(i) Draw $\overline{BC} = 5cm$.

(ii) Draw an arc of radius 6cm with centre C and another arc of radius 5cm with centre B cutting first in A.

(iii) Through A draw a line $\ell \parallel BC$.

(iv) Draw the $\perp$bisector of BC cutting the line ℓ in E.

(v) Draw CF $\perp$ on ℓ. CDEF is the rectangle.

(vi) Produce $\overline{DE}$ and cut $\overline{EL} = \overline{EF}$, bisect $\overline{DL}$ at O.

(vii) Draw a semicircle with centre O and radius $\overline{OL} = \overline{OD}$, cutting l in M.

(viii) Draw a square EMNR with side EM.

This is the required square equal in area to ΔABC.

OBJECTIVE

1. A triangle having two sides congruent is called: ____

(a) Scalene (b) Right angled
(c) Equilateral (d) Isosceles

2. A quadrilateral having each angle equal to 90° is called ____

(a)Parallelogram (b)Rectangle
(c)Trapezium (d) Rhombus

3. The right bisectors of the three sides of a triangle are ____

(a)Congruent (b) Collinear
(c)Concurrent (d) Parallel

4. The __ altitudes of an isosceles triangle are congruent:

(a)Two (b) Three
(c)Four (d) None

5. A point equidistant from the end points of a line segment is on its __
(a)Bisector (b)Right bisector
(c)Perpendicula (d) Median

6. ___ congruent triangles can be made by joining the mid points of the sides of a triangle:
(a)Three (b) Four
(c)Five (d) Two

7. The diagonals of a parallelogram ___ each other:
(a)Bisect (b) Trisect
(c)Bisect at right angle
(d) None of these

8. The median of a triangle cut each other in the ratio:
(a)4:1 (b) 3:1
(c)2:1 (d) 1:1

9. One angle on the base of an isosceles triangle is 30°. What is the measure of its vertical angle:
(a)30° (b) 60°
(c)90° (d) 120°

10. If the three altitudes of a triangle are congruent then the triangle is __
(a)Equilateral (b) Right angled
(c)Isosceles (d) Acute angled

11. If two medians of a triangle are congruent then the triangle will be:__
(a)Isosceles (b) Equilateral
(c)Right angled (d) Acute angled

12. A line segment joining a vertex of a triangle to the midpoint of its opposite side is called a ___ of the triangle:
(a)Altitude (b) Median
(c)Angle bisector (d) Right bisector

13. A line segment from a vertex of triangle perpendicular to the line containing the opposite side, is called an __ of the triangle:
(a)Altitude (b) Median
(c)Angle bisector (d) Right bisector

14. The point of concurrency of the three altitudes of a Δ is called its ___
(a)Ortho centre (b) In centre
(c)Circum centre (d) None

15. The internal bisector of the angle of a triangle meet at a point called the ______ of the triangle:
(a)In centre (b) Ortho centre
(c)Circum centre (c) None

16. The point of concurrency of the three perpendicular bisectors of the sides of a triangle is called the ____ of the triangle.
(a) Circum centre (b) In centre
(c) Ortho centre (d) None

ANSWER KEY

1.	d	2.	b	3.	c	4.	a	5.	b
6.	b	7.	a	8.	c	9.	d	10.	a
11.	a	12.	b	13.	a	14.	a	15.	a
16.	a								